Università degli Studi di Pisa

Dipartimento di Matematica

Tesi di Laurea Magistrale

Configurations of Kodaira Fibers on Elliptic Surfaces

8 giugno 2018

Candidato:
Giacomo Mezzedimi

Relatrice:
Prof.ssa Rita Pardini

Controrelatore:
Dott. Davide Lombardo

Anno Accademico 2017/2018

Contents

Introduction

The study of algebraic surfaces has always been a central field in Algebraic Geometry: since the Italian geometers, Castelnuovo and Enriques above all, started at the end of 19th century the impressive theory known today as the *Enriques classification* of surfaces, it has never stopped to ask mathematicians new fascinating questions. Following the modern terminology, we can coarsely divide algebraic surfaces into 4 big classes, depending on the asymptotic behaviour of the plurigenera P_n, and identified by the *Kodaira dimension* $\kappa \in \{-\infty, 0, 1, 2\}$. Although the early geometers had a good understanding of the first two classes (namely, of the surfaces with all $P_n \leq 1$), they didn't know much about those with $\kappa = 1$, except that they admitted an *elliptic fibration*, that is, a fibration onto a curve with almost all fibers smooth of genus 1; these surfaces are called *elliptic*. It wasn't until the sixties, that Kodaira provided an exhaustive study of them: he classified the possible singular fibers and he introduced several fundamental tools, as the monodromy around singular fibers and the Mordell-Weil group of sections.

Nowadays elliptic surfaces are quite well understood, but they are a neverending source of examples: for instance, many rational, K3 and Enriques surfaces admit an elliptic fibration, and they can be studied by the means of elliptic surfaces. They play a fundamental role in many geometric and arithmetic problems, and although they do not require too much prerequisites, they allow many applications of distinct fields of mathematics, such as graph theory, lattice theory, and the theory of singularities.

In this thesis we deal with *Jacobian* elliptic surfaces, i.e. elliptic surfaces admitting a section. This makes our treatment much less general, but it allows us to explain all the main geometric features of elliptic surfaces and to present the Mordell-Weil group of sections, from which we derive several fundamental consequences. In particular we focus on a problem investigated by Beauville, Miranda and Persson (see [Bea82], [MP86], [Per90], [Mir90], [MP89]): the classification of possible configurations of singular fibers on elliptic surfaces. In other words, given a list of Kodaira fibers, is there an elliptic surface with exactly those singular fibers? We only limit ourselves to the manageable cases, namely the rational and K3 cases, as the others admit an insanely high number of possible configurations; however, most of the techniques we introduce are general, and they could be used to study the problem in much more complex situations.

Since the number of necessary prerequisites is relatively small, we have tried to make the exposition as much as self-contained as possible; moreover, we provide appropriate references when we need a hard result whose proof goes beyond the scope of this thesis. Obviously, we take for granted all we consider "basic" in Algebraic Geometry, as sheaf theory, Hodge theory and the first notions of algebraic surfaces. For a detailed exposition and the complete proofs, we refer to the classics, such as [Har77], [BPV84] and [Bea96]. Now let us briefly present the content of the chapters.

The first chapter contains all the prerequisites: we start by recalling the basics of elliptic curves, and we introduce the Dynkin diagrams. Afterwards, we provide an exhaustive treatment of *simple* (or *du Val*) singularities on surfaces: we characterize them as the double covers branched along *A*-*D*-*E* curves, as the rational double points arising after the contraction of an *A*-*D*-*E* curve, and as the singularities that do not affect adjunction. At the end, we study the root systems corresponding to Dynkin diagrams, and we compute their discriminant form groups.

The second chapter is the central core of the thesis: it includes all the standard theory of elliptic surfaces. For this reason it is particularly dense, hence we prefer to describe the content of each section individually.

- In the first section we give the first definitions, we classify the possible singular fibers, and we show many examples of rational elliptic surfaces arising from the resolution of pencils of generically smooth plane cubics, to prove that the possible singular fibers can actually occur. We conclude the section by proving the uniqueness of the smooth minimal elliptic model.

- The second section deals with *Weierstrass fibrations*, i.e. fibrations $\pi\colon X \to C$ onto a smooth curve with irreducible fibers of genus 1 and with a smooth elliptic curve as generic fiber. This forces the existence of a distinguished "zero" section, given by the closure of the origin of the generic fiber. We pass from a smooth elliptic surface to a Weierstrass fibration simply contracting all the components of reducible fibers not meeting the zero section, and vice versa taking the unique smooth minimal elliptic model. We derive a global equation for a Weierstrass fibration X of the form $y^2 = x^3 + A(t)x + B(t)$, where A and B are sections of appropriate powers of the *fundamental line bundle* $\mathscr{L}$ on the base curve C, and we show the two standard representations of X: the representation as a divisor in a $\mathbb{P}^2$-bundle over C and as a double covering of a ruled surface R branched along a trisection T and a section of R. We notice that the type of singular fiber on X over $c \in C$ is completely determined by the local behaviour of the trisection T and the corresponding fiber of R, and this allows us to classify the singular fibers according to three simple numbers: the local orders of vanishing a, b, δ of the three sections $A, B, \Delta = 4A^4 + 27B^2$. This classification has several interesting consequences: for instance, we completely determine the ramification of the map $j\colon C \to \mathbb{P}^1$ associating to each $c \in C$ the j-invariant of the fiber X_c over c.

- The third section computes all the standard invariants of an elliptic surface, as the irregularity and the plurigenera, and classifies elliptic surfaces according to the genus of the base curve and the degree of the fundamental line bundle $\mathscr{L}$.

- In the fourth section we introduce a useful tool to deform elliptic surfaces, the quadratic twists. These allow us to interchange "$*$-fibers" with "non-$*$-fibers" preserving the j-map, and they are an important machinery to construct elliptic surfaces with prescribed configuration of singular fibers.

- In the fifth section we present a fundamental classification of germs of singular fibers, and we study the effect of base changes on them: in particular we see that some of them become smooth after a base change of finite order, hence they can be realized as quotients of smooth germs. Using these explicit descriptions we compute the monodromy around singular fibers, and we see that it only depends on the type of singular fiber.

- In the last section we study the Mordell-Weil group of sections: we prove that it is a finitely generated abelian group and we relate its rank to the rank of the Néron-Severi group by the Shioda-Tate formula. Moreover we compute the root lattice and the discriminant form group determined by each singular fiber, and we show that torsion sections are completely identified by the components of the fibers they meet.

Finally, in the last chapter we investigate the problem discussed above of the possible configurations of singular fibers. We start by studying the very special case of *extremal* rational elliptic surfaces, i.e. rational elliptic surfaces with a finite number of sections. The low number of possible configurations allows us to be extremely concrete: we prove the existence of the configurations providing pencils of

plane cubics generating rational elliptic surfaces with the prescribed singular fibers. In this way we obtain explicit equations for the surfaces, and we show the beautiful geometry lying underneath the theory of elliptic surfaces.
However, in order to extend the results to all rational elliptic surfaces, we have to change approach, as the number of configurations becomes unmanageable by hand. Therefore we introduce a combinatorial technique to prove the existence of the possible configurations, which uses the classification of branched coverings of $\mathbb{P}^1$. This let us reduce the original problem to deciding the existence of certain permutations, which can be easily done by a computer.
At the end, we study the analogous problem for *semistable* elliptic K3 surfaces, i.e. elliptic K3 surfaces admitting only I_n singular fibers. The existence of the possible configurations closely follows the strategy introduced for rational surfaces, while the impossibility of the other configurations is proved using some *ad hoc* results on elliptic K3 surfaces concerning their quotients by torsion sections.

Chapter 1

Preliminaries

1.1 Elliptic Curves

In this first introductory section we recall some basic and well known facts about elliptic curves: this will help us fix some notations and ease some computations later on. For a complete exposition, we refer to [Sil09]. For the moment, let K be a field of characteristic different from 2 and 3.

An *elliptic curve* over K is a pair (E, O), where E is a complete curve of genus 1 defined over K, i.e. satisfying a polynomial equation with coefficients in K, and $O \in E$ is a given point defined over K. Since in most cases O is the neutral element of the group law on the K-rational points of E, we will refer to the elliptic curve simply as E.

There are many instructive examples of elliptic curves; we just recall the most important ones.

Example 1.1.1 (Elliptic curves over $\mathbb{C}$). Fix a complex number τ with positive imaginary part. Then $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ is a maximal lattice in $\mathbb{C}$, and the quotient $E = \mathbb{C}/\Lambda$ is an elliptic curve. Obviously, the origin O of the group law on E is the class of the neutral element $0 \in \mathbb{C}$. This is not surprising: an elliptic curve is topologically a torus of real dimension 2, just as the quotient $\mathbb{C}/\Lambda$.

Example 1.1.2. If K is algebraically closed, every smooth cubic curve $E \subseteq \mathbb{P}^2$ can be made into an elliptic curve over K by choosing the origin O to be a flex point of the plane cubic.

Example 1.1.3. Let E be the double cover of $\mathbb{P}^1$ branched over ∞, p_1, p_2, p_3, where p_1, p_2, p_3 are points defined over K. Then Hurwitz's formula says that (E, O) is an elliptic curve, where O is the point over ∞. It is easy to prove that E is isomorphic to the plane cubic $y^2 = (x - p_1)(x - p_2)(x - p_3)$.
Another interesting class of elliptic curves is given by the plane cubics of the form $y^2 = x^3 + Ax + B$, where A, B are numbers in K. An equation of this form is said a *Weierstrass equation*, and a simple computation gives that such an elliptic curve is nonsingular if and only if the *discriminant*

$$\Delta = 4A^3 + 27B^2$$

is non-zero.

Lemma 1.1.4. *Every smooth elliptic curve E over K is isomorphic to a plane cubic given by a Weierstrass equation, with origin at the point at infinity.*

Proof. Consider the vector spaces $V_n = H^0(\mathcal{O}_E(nO))$ for $n \geq 0$. By Riemann-Roch (see [Ser97, p. 27] for a proof valid for arbitrary fields), we have that the dimension of V_n is exactly n. Hence, we can fix bases: $\{1\}$ for V_1, $\{1, f\}$ for V_2, $\{1, f, g\}$ for V_3, $\{1, f, g, f^2\}$ for V_4, $\{1, f, g, f^2, fg\}$ for V_5 and $\{1, f, g, f^2, fg, f^3\}$ for V_6. Notice that all these elements are linearly independent since they have a pole of different orders. However, $g^2 \in V_6$, and so there must exist a non-trivial linear relation

$$g^2 = a_6 f^3 + a_5 fg + a_4 f^2 + a_3 g + a_2 f + a_0,$$

for some $a_i \in K$. By scaling f and g appropriately, we can assume $a_6 = 1$; by completing the square in g we can assume $a_3 = a_5 = 0$, and by completing the cube in f we can assume $a_4 = 0$. This gives the desired Weierstrass equation. $\square$

A natural question is whether there exist other pairs $(A', B') \in K \times K$ such that the elliptic curves given by the equations $y^2 = x^3 + Ax + B$ and $y^2 = x^3 + A'x + B'$ are isomorphic. Thanks to Lemma 1.1.4, it is immediate to see that these two elliptic curves are isomorphic if and only if $A' = \lambda^4 A$ and $B' = \lambda^6 B$ for some $\lambda \in K^*$, since $(x', y') = (\lambda^2 x, \lambda^3 y)$ is the only change of variables that preserves the form of the Weierstrass equation.

Therefore the quantity $j(A, B) = j(E) = \frac{4A^3}{4A^3+27B^2} = \frac{4A^3}{\Delta} \in K$ is invariant under the action of K^* on $K \times K$ given by $\lambda \cdot (A, B) = (\lambda^4 A, \lambda^6 B)$, and consequently isomorphic elliptic curves have the same $j(E)$. For this reason, this number in K is called the j-invariant of the elliptic curve. Actually, our definition of the j-invariant differs from the usual one by a factor 1728; however, this does not change much, thanks to our assumption $\mathrm{char}(K) \neq 2, 3$. Sometimes we will say that the j-invariant of E is the element $[4A^3, \Delta] \in \mathbb{P}^1(K)$.

We can notice that $j(A, B) = 1$ if and only if $B = 0$, and $j(A, B) = 0$ if and only if $A = 0$. In both cases we get a unique smooth elliptic curve (up to isomorphism over $\overline{K}$). This is always the case, as the next result shows:

Lemma 1.1.5. *If $j(A, B) = j(A', B')$, then the elliptic curves given by (A, B) and (A', B') are isomorphic over $\overline{K}$.*

Proof. We can assume $j(A, B) = j(A', B') \neq 0, 1$ by what we have just said. This implies that $A^3 B'^2 = A'^3 B^2$. Take $\gamma = \frac{A'B}{AB'} \in K$, and let λ be a square root of γ in $\overline{K}$. Then it is immediate to see that the change of variables $(x', y') = (\lambda^2 x, \lambda^3 y)$ gives an isomorphism of the two curves over $K(\lambda)$. $\square$

This would be sufficient if we were only interested in smooth elliptic curves. However, the study of elliptic surfaces requires a good knowledge also of singular elliptic curves, because often smooth fibers degenerate and create singularities. Consequently, we will need a generalization of Lemma 1.1.4, provided by the next theorem.

Let E be a reduced irreducible complete curve of arithmetic genus 1 over K, together with a smooth closed point $O \in E$. Similarly to the smooth case, consider the vector spaces $V_n = H^0(\mathcal{O}_E(nO))$; again, by Riemann Roch, V_n has dimension n. Analytically, V_n is the space of rational functions with exactly one pole at O, of order at most n (if $n \geq 2$); $V_0 = V_1$ is the space of constant functions on E. Thus we can consider each V_i as a subspace of V_{i+1}, and multiplication gives a well defined map

$$P_i^k \colon \mathrm{Sym}^k V_i \longrightarrow V_{ki}$$

for every $k, i \geq 0$.

Lemma 1.1.6. *1. There exists $y \in V_3 \backslash V_2$ such that y^2 is in the image of P_2^3.*

2. There exists $x \in V_2 \backslash V_1$ such that $y^2 = x^3 + Ax + b$ for some $A, B \in K$.

3. If (x, y) and (x', y') satisfy these two conditions, then there exists $\lambda \in K^*$ such that $x' = \lambda^2 x$ and $y' = \lambda^3 y$.

Proof. Points 1. and 2. are proved exactly as in Lemma 1.1.4. To prove the last assertion, write $y' = \alpha y + \beta x + \gamma$ and $x' = ax + b$, with $a, \alpha \neq 0$. Then

$$(y')^2 = \alpha^2 y^2 + 2(\beta x + \gamma)y + (\beta x + \gamma)^2 = \alpha^2(x^3 + Ax + B) + 2(\beta x + \gamma)y + (\beta x + \gamma)^2,$$

and since $(y')^2$ belongs to the image of P_2^3, necessarily the second term on the right hand side must be 0, i.e. $\beta = \gamma = 0$. Hence $y' = \alpha y$. Now

$$(x')^3 + A'x' + B' = (y')^2 = \alpha^2 y^2 = \alpha^2(x^3 + Ax + B),$$

and since on the right hand side there is no x^2 term, b must be 0. Substituting the equalities $y' = \alpha y$ and $x' = ax$, we find that $\alpha^2 = a^3 \in K$, as desired. □

Theorem 1.1.7. *Let E be a reduced irreducible complete curve of arithmetic genus 1 over K. There exist $A, B \in K$ such that E is isomorphic to the plane cubic $y^2 = x^3 + Ax + B$. The pair (A, B) is unique up to the action of K^* on $K \times K$ given by $\lambda \cdot (A, B) = (\lambda^4 A, \lambda^6 B)$.*
The pair (x, y) is said a Weierstrass basis for the curve E.

Proof. We only have to show that $y^2 = x^3 + Ax + B$ is the only relation between x, y. The divisor corresponding to the point O is ample, so the affine curve E is isomorphic to the affine scheme $\operatorname{Spec}(R)$, where $R = \bigcup_{n \geq 0} V_n$. It is clear that $\{1, x, \ldots, x^m, y, xy, \ldots, x^{m-1}y\}$ is a basis for V_{2m+1} for every $m \geq 0$, and since the relation $y^2 = x^3 + Ax + B$ gives the correct Hilbert polynomial $\dim R_{\leq n} = n$, we obtain that

$$R \cong \frac{K[x, y]}{(y^2 = x^3 + Ax + B)},$$

as wanted. The uniqueness is quite easy using part 3. of the previous lemma, and we leave it to the reader. □

To sum up the results, we have that every elliptic curve E is isomorphic to a plane cubic in Weierstrass form. If the discriminant Δ is non-zero, E is smooth; otherwise, there are two possibilities:

- $A = B = 0$, and E is a cuspidal rational curve;
- $A, B \neq 0$, so (A, B) is in the orbit of $(-3, 2)$, and E is a nodal rational curve.

1.2 Graph Theory

Let G be a *graph*, i.e. a pair (V, E), where V is an ordered set of vertices and E is a set of edges. We will accept that G has loops and multiple edges.

Consider the $\mathbb{Q}$-vector space V_G generated by the vertices V. We can define a symmetric bilinear form on V_G by imposing that, for every $v \neq w \in V$, the following hold:

- $v^2 = -2 + 2 \cdot \#\{\text{loops at } v \text{ in } E\}$;
- $vw = \#\{\text{edges between } v \text{ and } w\}$.

This form on V_G is called the *associated form* of G, and it will be crucial for our study.

We are now going to introduce a special class of connected graphs, called (extended) Dynkin diagrams, that we list in Tables 1.1 and 1.2. Later we will understand their importance, and how they naturally arise in the study of elliptic surfaces.

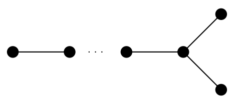
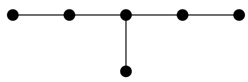
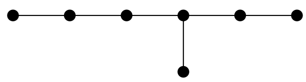
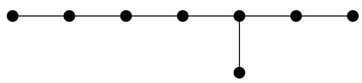
Name	**Dynkin diagram**
A_n, $n \geq 1$	(there are n vertices)
D_n, $n \geq 4$,	 (there are n vertices)
E_6	
E_7	
E_8	

Table 1.1: Dynkin diagrams.

Now let G be a Dynkin diagram, and consider the vector space V_G endowed with the associated form.

Lemma 1.2.1. *If G is a Dynkin diagram, the associated form on V_G is negative definite.*

Proof. There are no loops in the Dynkin diagrams, so $v^2 = -2$ for each $v \in V_G$. Assume G is of type A_n. Given $\sum_{i=1}^{n} \alpha_i v_i \in V_G$, with at least one $\alpha_i \neq 0$, we have

$$\left(\sum_{i=1}^{n} \alpha_i v_i\right)^2 = -2\sum_{i=1}^{n} \alpha_i^2 + 2\sum_{i=1}^{n-1} \alpha_i \alpha_{i+1} \leq -\alpha_1^2 - \alpha_n^2 \leq 0,$$

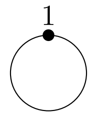
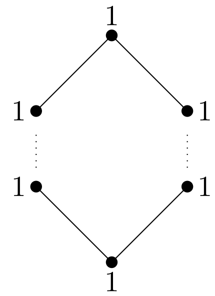
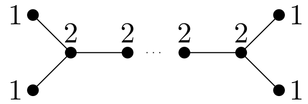
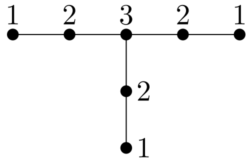
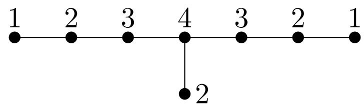
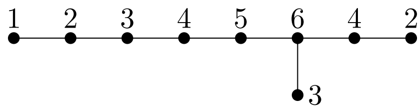
Name	**Extended Dynkin diagram**
$\widetilde{A}_0$	
$\widetilde{A}_n$, $n \geq 1$	 (there are $n+1$ vertices)
$\widetilde{D}_n$, $n \geq 4$	 (there are $n+1$ vertices)
$\widetilde{E}_6$	
$\widetilde{E}_7$	
$\widetilde{E}_8$	

Table 1.2: Extended Dynkin diagrams. The weights attached to each vertex are explained below.

using that vertex v_i is only connected with vertices $v_{i\pm1}$, and the simple inequality $\alpha_i^2+\alpha_{i+1}^2 \geq 2\alpha_i\alpha_{i+1}$. But here equality holds if and only if $\alpha_i = \alpha_{i+1} = 0$, hence the equalities above hold if and only if all the α_i's are 0.
Now the other verifications are easy: for instance, if G is of type D_n, it contains A_{n-2} as a subgraph, so any element in V_G can be written as $S + \alpha v + \beta w$, with $S = \sum_{i=1}^{n-2} \alpha_i v_i \in V_{A_{n-2}}$, and v, w are the last two vertices. Therefore

$$(S + \alpha v + \beta w)^2 \leq -\alpha_{n-2}^2 - 2\alpha^2 - 2\beta^2 + 2\alpha\alpha_{n-2} + 2\beta\alpha_{n-2},$$

and we conclude as above. We leave the remaining computations to the reader. □

This simple lemma has an important consequence: since every extended Dynkin diagram G contains a Dynkin diagram as a subgraph, we have that the space V_G has a negative definite subspace of codimension 1. The next lemma completes the study of the associated form on V_G.

Lemma 1.2.2. *If G is an extended Dynkin diagram, the associated form on V_G is negative semidefinite, with a 1-dimensional kernel generated by the element $X_0 \in V_G$ whose coefficients in the basis of the vertices are given in Table 1.2.*

Proof. As noticed, thanks to Lemma 1.2.1 we only need to show that X_0 is in the kernel of the form. Equivalently, it suffices to show that $X_0 v = 0$ for each $v \in V$. This is an easy computation, and we leave it to the reader. □

The importance of the (extended) Dynkin diagrams is revealed by the next surprising result:

Lemma 1.2.3. *1. Every connected graph G either is contained in or contains an extended Dynkin diagram.*

2. Every connected graph G without loops or multiple edges either is contained in or contains an extended Dynkin diagram without loops or multiple edges (i.e., not $\widetilde{A}_0$ or $\widetilde{A}_1$).

Proof. If a graph has a loop or a multiple edge, it contains $\widetilde{A}_0$ or $\widetilde{A}_1$. Thus the two parts of the lemma are equivalent, and we decide to show the second.
If G contains a n-cycle, then it contains $\widetilde{A}_n$, so we can assume that G has no cycles (such a graph is called a *tree*). Moreover, if G contains a vertex of degree ≥ 4 (the *degree* is simply the number of edges emanating from the vertex), then it contains $\widetilde{D}_4$, and so we can assume that the degree of each vertex is at most 3.
If G has two vertices of degree 3, it contains $\widetilde{D}_{n+4}$, where n is the length of a path connecting the two vertices (that exists because G is connected); from the same hypothesis we obtain that if all vertices of G have degree 1 or 2, then G is contained in some $\widetilde{A}_n$. Summing up all these remarks, we can assume that G has exactly one vertex of order 3, and if p, q, r are the lengths of the 3 paths emanating from this vertex (counting the central vertex itself), we will denote G by $T_{p,q,r}$: clearly these 3 numbers identify the graph G. Just for the sake of clarity, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ are respectively $T_{3,3,3}$, $T_{2,4,4}$ and $T_{2,3,6}$ with this notation.
Since the order of the p, q, r doesn't affect G, we impose that $2 \leq p \leq q \leq r$. If $p \geq 3$, G contains $\widetilde{E}_6$, so assume $p = 2$. If $q \geq 4$, G contains $\widetilde{E}_7$, while if $q = 2$, G is contained in some $\widetilde{D}_n$, so assume $q = 3$. Finally, if $r \leq 4$, G is contained in $\widetilde{E}_7$, and if $r \geq 5$, G contains $\widetilde{E}_8$. □

We can refine the result of the previous lemma imposing the condition that the associated form on V_G is negative semidefinite, with a 1-dimensional kernel: the next theorem provides a fundamental characterization of the extended Dynkin diagrams.

Theorem 1.2.4. *If G is a connected graph, whose associated form is negative semidefinite with a 1-dimensional kernel, then G is an extended Dynkin diagram.*

Proof. First, let's work out the case when G has a loop or a multiple edge. If G has a loop at v, then $v^2 \geq 0$, but the negative semidefiniteness forces $v^2 = 0$; thus, if there exists an edge between v and $w \neq v$, we have $(v + w)^2 = 2vw + w^2 \geq 0$, and again this must be an equality, so $w^2 = -2$ and $vw = 1$. But this would imply $(2v + w)^2 = 2$, a contradiction, and so G is simply $\widetilde{A}_0$. If instead G has a multiple edge between v and w, surely it can have no loops, and since $(v + w)^2 \geq -4 + 2 \cdot 2 = 0$, we need to have an equality. If z is another vertex, then the connectedness of the graph implies that $(v + w + z)^2 \geq z^2 + (v + w)^2 + 2z(v + w) \geq 0$, and as before we get a contradiction, forcing G to be $\widetilde{A}_1$. Now we can turn to the case when G has no loops or multiple edges.
Thanks to Lemma 1.2.3 we can assume that G is contained in or contains an extended Dynkin diagram different from $\widetilde{A}_0$ and $\widetilde{A}_1$; in the first case, if G is strictly contained in the extended Dynkin diagram, then it is a Dynkin diagram, whose associated form is negative definite, a contradiction. In the second case, an argument as before shows that the class X_0 must span the kernel of the form on V_G, and so G coincides with the Dynkin diagram: if there existed a vertex v outside the Dynkin diagram, then $X_0 v = 0$, hence v would not be connected with the Dynkin diagram, since the weights in X_0 are all positive and v is not contained inside X_0. □

Similarly we have:

Theorem 1.2.5. *If G is a connected graph, whose associated form is negative definite, then G is a Dynkin diagram.*

Proof. An argument as above shows that G cannot have a loop or a multiple edge. If G strictly contains an extended Dynkin diagram D with kernel spanned by X_0, then for every vertex $v \in G \backslash D$ we have $0 > (X_0 + v)^2 = 2X_0 v + v^2 = 2X_0 v - 2$, hence $X_0 v = 0$, a contradiction since the associated form is negative definite on G. Then G is strictly contained in an extended Dynkin diagram, thus it is a Dynkin diagram. $\square$

1.3 Simple Singularities on Surfaces

In the whole discussion we will often deal with singularities of surfaces, and we will need to study how to resolve them. We collect in this preliminary section all the results we will need, in order to give a clearer exposition later on. In the exposition we will follow [BPV84, Sections II.8 and III.7], [Rei96, Chapter 4]; for the original work by du Val, see [du 34]. From now on, we will work over the field $\mathbb{C}$ of complex numbers.

First of all, we have to specify what *simple* singularities mean. Let X be a smooth surface, and $C = \{f = 0\}$ be a reduced curve in it. We begin with a general result concerning the reduced total transform of C. Throughout the exposition, we will denote by $g(C)$ the arithmetic genus of C.

Lemma 1.3.1. *There exists a resolution of singularities $\eta\colon Y \to X$ of C such that its strict transform $\widetilde{C}$ is smooth.*

Proof. Assume that C is irreducible. If C is smooth, we are done; otherwise, let $p \in C$ be a singular point for C. We blow up X at p_1, and we obtain a map $\epsilon_1\colon X_1 \to X$; if the strict transform C_1 of C inside X_1 is smooth, again we are done; otherwise we repeat the same argument. We only have to show that the process ends. Assume by contradiction that we have an infinite sequence of blow-ups $\epsilon_k\colon X_k \to X_{k-1}$ of singular points $p_k \in X_{k-1}$, with $X_0 = X$, and denote by C_k the strict transform of C inside X_k; then by the genus formula

$$g(C_k) = 1 + \frac{C_k^2 + C_k K_{X_k}}{2} = 1 + \frac{(\epsilon_k^* C_{k-1} - n_k E_k)^2 + (\epsilon_k^* C_{k-1} - n_k E_k)(\epsilon_k^* K_{X_{k-1}} + E_k)}{2} =$$
$$= 1 + \frac{C_{k-1}^2 + C_{k-1} K_{X_{k-1}}}{2} - \frac{n_k^2 - n_k}{2} = g(C_{k-1}) - \frac{n_k^2 - n_k}{2}$$

where E_k is the exceptional divisor over p_k, and $n_k > 1$ is the multiplicity of C_{k-1} at p_k. But this is absurd, since the numbers $n_k^2 - n_k$ are greater or equal than 2, and the genus $g(C_k)$ cannot be negative.
If instead C is reducible, we apply the previous argument to each component of C; then the only singular points of the strict transform come from the intersection points of the components. Since these intersection are finitely many, we can just blow up each of them an appropriate number of times (precisely, as many times as their intersection number) and obtain a smooth strict transform of C. $\square$

Proposition 1.3.2. *There exists a resolution of singularities $\tau\colon Y \to X$ of C such that its reduced total transform $\overline{C}$ has at worst nodal singularities.*

Proof. Let $\eta\colon X' \to X$ be a map such that the strict transform $\widetilde{C}$ of C is nonsingular. If E is the union of all the exceptional curves for η, then the reduced total transform of C is $\eta^{-1}(C) = \widetilde{C} \cup E$. Since $\widetilde{C}$ and all the components of E are smooth, we can just blow up the singularities of $\eta^{-1}(C)$ with a map $\eta'\colon Y \to X'$; the singularities on the reduced total transform $\overline{C} = (\eta')^{-1}(\eta^{-1}(C))$ are transverse, since all the components of $\eta^{-1}(C)$ are smooth. Hence we only have to check that no triple point occurs on $\overline{C}$. But if this were the case, a component $C_i \subseteq \eta^{-1}(C)$ would intersect two exceptional divisors for η', and so it would have a singularity there, a contradiction. $\square$

Definition 1.3.3. A point $c \in C$ is said to be a *simple curve singularity* if it is a double or triple point such that, when resolving the singularity to a collection of nodes according to Proposition 1.3.2, after each blow-up, the reduced total transform of C has again only double or triple points.

Notice that we are not considering the *triple tacnodes* (or $[3,3]$ *points*), i.e. triple points that remain triple in the strict transform after the blow-up. Indeed, a triple tacnode looks locally as in Figure 1.1, and, when we blow it up, the reduced total transform acquires a quadruple point.

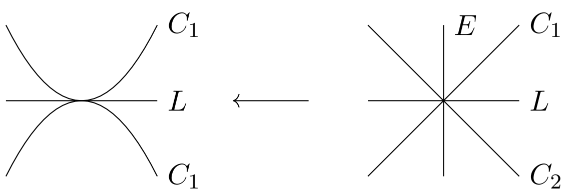


Figure 1.1: Blow up of a triple tacnode.

We choose local coordinates x, y around c such that c corresponds to $(x, y) = (0, 0) \in X$.

Classification of double points Let $c \in C$ be a double point, i.e. a point with multiplicity 2. There are two possibilities, that correspond to whether c becomes a simple point or it remains double after blowing up. After a linear transformation, these two cases correspond to whether

$$f_2 = x^2 + y^2 \qquad \text{or} \qquad f_2 = x^2,$$

where f_2 is the residue class of $f \pmod{\mathfrak{m}^3}$, $\mathfrak{m} \subseteq \mathcal{O}_{(0,0)}$ being the maximal ideal.
The first case is fairly clear: we can find holomorphic functions φ_1, φ_2 such that $f = x^2\varphi_1(x, y) + y^2\varphi_2(x, y)$, with $\varphi_1(0, 0), \varphi_2(0, 0) \neq 0$; hence we can change variables $x = x\sqrt{\varphi_1}$ and $y = y\sqrt{\varphi_2}$ in a sufficiently small neighbourhood of $(0, 0)$ and get the *normal form*

$$f = x^2 + y^2.$$

In other words, c is a node, and we will say that c is of type A_1.
The second case is a bit more difficult, and it depends on the number $n = \dim \frac{\mathcal{O}_{X,c}}{(f_x, f_y)}$, called the *Milnor number*. Notice that this number is finite: otherwise, the set $\{f_x = f_y = 0\}$ would contain a curve passing through c, and f would vanish on this curve, implying that C is not reduced. We can write

$$f(x, y) = x^2 e(x, y) + x\varphi(y) + \psi(y),$$

where e, φ, ψ are holomorphic functions such that $e(0, 0) \neq 0$, and φ, ψ vanish at $y = 0$ with order respectively $k \geq 2$ and $l \geq 3$. Up to a unit $\varphi = y\varphi_y$, and so

$$(f_x, f_y) = (2xe + x^2 e_x + \varphi, x^2 e_y + x\varphi_y + \psi_y) \subseteq (x, \varphi_y, \psi_y),$$

from which $\min\{k, l\} \leq n + 1$. In a sufficiently small neighbourhood of $(0, 0)$ we can change variable $x = x\sqrt{e} + \frac{\varphi}{2\sqrt{e}}$ and obtain

$$f = x^2 - \frac{1}{4e}\varphi^2 + \psi.$$

Again, if we expand $-\frac{1}{4e} = c(y) + x\varphi_1(y) + x^2 g(x, y)$ and put $e' = 1 + \varphi^2 g$, $\varphi' = \varphi^2\varphi_1$, $\psi' = \psi + \varphi^2 c$, we get an expression for f

$$f(x, y) = x^2 e'(x, y) + x\varphi'(y) + \psi'(y)$$

analogous to the first one, but where the order of vanishing of φ' is at least $2k$. Therefore, since we have the inequality $\min\{k, l\} \leq n + 1$, we can assume that $k \geq l$ and write $x\varphi(y) + \psi(y) = y^l e''(x, y)$, with of course $e''(0, 0) \neq 0$. After the last change of variables $x = x\sqrt{e}$ and $y = y\sqrt[l]{e''}$, we obtain a normal form

$$f = x^2 + y^l,$$

and from the hypothesis $n = \dim \frac{\mathcal{O}_{X,c}}{(f_x, f_y)}$ we get $l = n + 1$. We can notice that if $n = 1$, the normal form of the singularity is exactly the form in the first case; hence it is natural to denote this singularity with the type A_n.

Classification of triple points with at least two different tangents Let's denote by f_3 the residue class of $f \pmod{\mathfrak{m}^4}$. The point c has multiplicity 3, and our hypothesis is that f_3 has at least two distinct roots. Thus there must be a simple root, that we can assume to correspond locally to the component $\{y = 0\}$. Moreover, since f is reduced, we necessarily have $f(x, y) = yh(x, y)$; the component $\{h(x, y) = 0\}$ has a double point at c, and so it has a normal form of one of the types described above. The residue class h_2 does not contain the factor y, so we can change coordinates around $(0, 0)$ (leaving the curve $\{y = 0\}$ invariant) and obtain a normal form

$$f = y(x^2 + y^{n-2})$$

for some $n \geq 4$. This triple point is said to be of type D_n.

Classification of triple points with one tangent only In this case, the residue class f_3 has one root only, that we can assume to be $x = 0$. Therefore we have an expansion

$$f(x, y) = x^3 e(x, y) + x^2 y^2 \varphi_1(y) + xy^3 \varphi_2(y) + y^4 \varphi_3(y),$$

with $e(0, 0) \neq 0$. The strict trasform $\widetilde{C}$ of C after the blow-up of c is given by the equation (we put $x = uy$ and we divide by y^3, i.e. we eliminate the term corresponding to the triple exceptional divisor)

$$\widetilde{f}(u, y) = u^3 e'(u, y) + u^2 y \varphi_1(y) + uy \varphi_2(y) + y \varphi_3(y).$$

Now $\widetilde{C}$ can have at worst a double point at $(0, 0)$ (the exceptional divisor passes through the point over c already), hence the residue class

$$(\widetilde{f})_2 = y\varphi_3(0) + uy\varphi_2(0) + y^2(\varphi_3)_y(0)$$

must not be identically zero. Thus we have three subcases, that we are going to study separately.

If $\varphi_3(0) \neq 0$, we can perform a change of variables $y = y\sqrt[4]{\varphi_3} + x\frac{\varphi_2}{4\sqrt[4]{\varphi_3^3}}$ and bring f to the form

$$f(x, y) = x^3 e(x, y) + x^2 y^2 \varphi_1(y) + y^4.$$

Now we set $x = x\sqrt[3]{e} + y^2 \frac{\varphi_1}{3\sqrt[3]{e^2}}$ and obtain an expression for f

$$f(x, y) = x^3 + y^4 e'(x, y),$$

for some $e'(0, 0) \neq 0$, and as usual we can reach the normal form

$$f = x^3 + y^4.$$

This singularity is said to be of type E_6.

If $\varphi_3(0) = 0$, but $\varphi_2(0) \neq 0$, then just by inspection of the residue class $(\widetilde{f})_2$ we see that the strict transform $\widetilde{C}$ has a node at c; hence the original triple point must be reducible, and if we choose the line $\{x = 0\}$ as one of the components, we can put f into the form

$$f(x, y) = x(x^2 e(x, y) + xy^2 \varphi_1(y) + y^3 \varphi_2(y)).$$

Since $\varphi_2(0) \neq 0$, we can change variable $y = y\sqrt[3]{\varphi_2} + x\frac{\varphi_1}{\sqrt[3]{\varphi_2^2}}$ and write f as

$$f(x, y) = x(x^2 e'(x, y) + y^3),$$

from which we reach the normal form

$$f = x(x^2 + y^3).$$

This singularity is said to be of type E_7.

Finally, if $\varphi_2(0) = \varphi_3(0) = 0$, but $(\varphi_3)_y(0) \neq 0$, then we can expand f into

$$f(x,y) = x^3 e(x,y) + x^2 y^2 \psi_1(y) + xy^4\psi_2(y) + y^5\psi_3(y),$$

with $\psi_3(0) \neq 0$. Substituting $x = x\sqrt[3]{e} + y^2 \frac{\psi_1}{3\sqrt[3]{e^2}}$, we obtain

$$f(x,y) = x^3 e(x,y) + x^2 y^4 \psi_1(y) + xy^4\psi_2(y) + y^5\psi_3(y),$$

where the e and the ψ_i are changed, but $\psi_3(0) \neq 0$ still holds. Changing the second variable $y = y\sqrt[5]{\psi_3} + x\frac{\psi_2}{5\sqrt[5]{\psi_3^4}}$ we obtain

$$f(x,y) = x^3 e(x,y) + x^2 y^3 \psi_1(y) + y^5,$$

from which the substitution $x = x^3\sqrt[3]{e} + y^3\frac{\psi_1}{3\sqrt[3]{e^2}}$ eliminates the central term, yielding at the end the normal form

$$f = x^3 + y^5.$$

This singularity is said to be of type E_8.

We can summarize this classification in the following result:

Theorem 1.3.4. *The only simple singularities occurring on a curve $C \subseteq X$ are exactly those of the forms*

Name	Normal form
A_n, $n \geq 1$	$x^2 + y^{n+1} = 0$
D_n, $n \geq 4$	$y(x^2 + y^{n-2}) = 0$
E_6	$x^3 + y^4 = 0$
E_7	$x(x^2 + y^3) = 0$
E_8	$x^3 + y^5 = 0$

Table 1.3: List of possible simple (A-D-E) singularities.

Proof. The previous work assures that all the simple singularities can be found in this list. We have to show the converse, i.e. that all these singularities are simple. Equivalently, we must show that the reduced total trasform of these curves have again only singularities of these types.

Clearly the total transform of A_1 has two nodes, so two singularities of type A_1. The total trasform of A_2 is

$$\overline{f} = y^2(u^2 + y),$$

but since we need the reduced total trasform we consider $\overline{f} = y(u^2 + y) = y^2 + u^2 y$. Its residue class modulo $\mathfrak{m}^3$ is non-zero, hence it has a double point at $(0,0)$. By our classification we only need to compute the Milnor number $n = \dim \frac{\mathcal{O}_{(0,0)}}{(\overline{f}_u, \overline{f}_y)}$. We have $\overline{f}_u = 2uy$, $\overline{f}_y = 2y + u^2$, so $(\overline{f}_u, \overline{f}_y)$ contains $uy, u^3 = u\overline{f}_y - \overline{f}_u$ and $y^2 = \frac{1}{2}y\overline{f}_y - \frac{1}{4}u\overline{f}_u$. We conclude that $\{1, y, u\}$ is a basis for $\frac{\mathcal{O}_{(0,0)}}{(\overline{f}_u, \overline{f}_y)}$, and so $\overline{f}$ has a singularity of type A_3.

For A_n, $n \geq 3$, the reduced total transform is

$$\overline{f} = y(u^2 + y^{n-1})$$

and this is already in normal form: the singularity is of type D_{n+1}.

For the D_n's we have to separate some cases. First of all, let's consider $n = 4$, i.e. $f = y(x^2 + y^2)$. The reduced total transform is

$$\overline{f} = ux(u^2 + 1) = ux(u + i)(u - i),$$

so we immediately see that we have 3 nodes. If instead $n = 5$, the normal form is $f = y(x^2 + y^3)$. We have already seen that the component $x^2 + y^3 = 0$ has a singularity of type A_3 in its reduced total transform; moreover, the component $y = 0$ intersects transversally this total transform, contributing to another node. It remains to study the case $n \geq 6$. Similarly to the case $n = 5$, we have a normal form $f = y(x^2 + y^{n-2})$, but with $n - 2 \geq 4$, and so the component $x^2 + y^{n-2} = 0$ is of type A_{n-3}, yielding a reduced total transform with singularity of type D_{n-2}. The component $\{y = 0\}$ adds another node.
The last three singularities E_6, E_7, E_8 have to be studied individually. The first one has a reduced total transform

$$\overline{f} = y(y + u^3) = y^2 + u^3 y,$$

and so it has a double point at $(0, 0)$. The ideal $(\overline{f}_u, \overline{f}_y)$ is generated by $2y + u^3$ and $u^2 y$, so $y^2, u^5 \in (\overline{f}_u, \overline{f}_y)$; it is immediate to see that $\{1, y, u, u^2, uy\}$ is a basis for $\frac{\mathcal{O}_{(0,0)}}{(\overline{f}_u, \overline{f}_y)}$, thus the singularity of the reduced total transform is A_5.
The singularity E_7 has reduced total transform

$$\overline{f} = uy(y + u^2) = u(y^2 + u^2 y),$$

and the second component $y^2 + u^2 y = 0$ has a double point at $(0, 0)$. In particular, we have already seen that this equation has a singularity of type A_3, i.e. it can be carried into the normal form $y^2 + u^4 = 0$, leaving invariant the line $u = 0$. Thus the reduced total transform has a normal form $\overline{f} = u(y^2 + u^4)$, hence it has a singularity of type D_6.
Finally, the singularity E_8 has reduced total transform

$$\overline{f} = y(u^3 + y^2),$$

i.e. $\overline{f} = 0$ has a singularity of type E_7. Overall, we can write all these informations in a table:

Singularity	**Singularity on the reduced total transform**
A_1	$2A_1$
A_2	A_3
A_n, $n \geq 3$	D_{n+1}
D_4	$3A_1$
D_5	A_1, A_3
D_n, $n \geq 6$	A_1, D_{n-2}
E_6	A_5
E_7	D_6
E_8	E_7

Table 1.4: Corresponding simple singularities after blow-up.

□

Corollary 1.3.5. *The simple curve singularities, or equivalently the A-D-E singularities, consist of the double points and the triple points that are not triple tacnodes.*

Now that we have completed the classification of the simple singularities on curves $C \subseteq X$, we can begin to focus on the singularities of the surface X itself. We will always deal with singularities on X arising from the contraction of a reduced curve $C \subseteq X$, and so we need to recall the following fundamental criterion, due to Grauert (see [Gra62]):

Theorem 1.3.6. *A reduced, compact and connected curve $C = \bigcup C_i$ on X is contractible if and only if the intersection matrix of the components C_i (i.e. the symmetric matrix M such that $M_{ij} = C_iC_j$) is negative definite.*

Remark 1.3.7. In the previous theorem, and in the following, we work with the analytic category. In other words, we will say that a curve $C \subseteq X$ is *contractible* (or *exceptional*) if there exists a *complex analytic* normal surface Y (not necessarily *algebraic*) and a proper analytic function $f\colon X \to Y$ contracting C to a point and inducing an isomorphism $X\backslash C \cong Y\backslash f(C)$. This will allow us to work locally, and with open varieties.

As an example, we know that every (-1)-curve on X, i.e. every smooth rational curve with self-intersection -1, is contractible, and moreover it is exceptional for a certain blow-up $\epsilon\colon X \to Y$. The first non-trivial example of reducible contractible curves is given by the so-called *A*-*D*-*E* curves, the ones we will be mostly interested in.

Definition 1.3.8. An *A*-*D*-*E* curve is a connected contractible curve $C = \bigcup C_i$ such that each component is a (-2)-curve, i.e. a smooth rational curve with self-intersection -2.

The contractibility hypothesis is strong: if C_i, C_j are two distinct components of an *A*-*D*-*E* curve C, we have $(C_i + C_j)^2 = -4 + 2C_iC_j$, thus the negative definiteness of the intersection matrix forces $C_iC_j \leq 1$. In other words, every pair of components of C do not intersect or they intersect transversally. Hence the dual graph of C (the graph with one vertex for each component C_i, such that the number of edges between C_i and C_j equals C_iC_j) is a connected graph with negative definite associated form, and so it is a Dynkin diagram. This explains the strange looking name of *A*-*D*-*E* curves. Obviously, we will say that C is an A_n, D_n, E_6, E_7, E_8 curve if its dual graph is a Dynkin diagram of that shape.

Clearly, if C is a connected contractible curve, then the map $\pi\colon X \to Y$ that contracts C to a point (possibly) creates a singularity only at $y = \pi(C) \in Y$. The best possible situation is when the singularity arising on Y is *rational*:

Definition 1.3.9. $y = \pi(C) \in Y$ is called a *rational* singularity if the sheaf $R^1\pi_*\mathcal{O}_X$ on Y vanishes.

Recall that the sheaf $R^1\pi_*\mathscr{F}$ is defined as the sheaf on Y associated to the presheaf $U \mapsto H^1(\pi^{-1}(U), \mathscr{F})$ for any coherent sheaf $\mathscr{F}$ on X. For a basic introduction, we refer to [BPV84, Section I.8].

We will need another result from [Gra62] to give a precise identification of rational singularities.

Theorem 1.3.10. *If $C \subseteq X$ is a contractible curve, then there exist arbitrarily small neighbourhoods U of C such that, for every locally free sheaf $\mathscr{F}$ of $\mathcal{O}_U$-modules, the restriction morphisms $H^i(U, \mathscr{F}) \to H^i(C, \mathscr{F}|_{kC})$ for all $i \geq 1$ are injective for $k \gg 0$.*

Remark 1.3.11. In particular, since C has dimension 1, the cohomology groups $H^2(C, \mathscr{F}|_{kC})$ vanish for every locally free sheaf $\mathscr{F}$, and so $H^2(U, \mathscr{F}) = 0$. Similarly $H^2(U, \mathscr{F}(-kC)) = 0$, and so just writing down the long exact sequence in cohomology we find that the restriction $H^1(U, \mathscr{F}) \to H^1(C, \mathscr{F}|_{kC})$ is even bijective for $k \gg 0$.

Proposition 1.3.12. *y is rational if and only if the dimensions $h^1(\mathcal{O}_{kC})$ are 0 for all $k \geq 1$.*

Proof. Fix $k \geq 1$, and consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-kC) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{kC} \longrightarrow 0.$$

$\mathcal{O}_{kC} = \mathcal{O}_X|_{kC}$ is the cokernel of the first inclusion, and we have a short exact sequence

$$0 \longrightarrow \frac{\mathcal{O}_X(-kC)}{\mathcal{O}_X(-(k+1)C)} \longrightarrow \mathcal{O}_{(k+1)C} \longrightarrow \mathcal{O}_{kC} \longrightarrow 0,$$

where the sheaf $\frac{\mathcal{O}_X(-kC)}{\mathcal{O}_X(-(k+1)C)}$ can be identified with $\mathcal{O}_C(-kC)$. This exact sequence is a special case of the so called *decomposition sequence*

$$0 \longrightarrow \mathcal{O}_A(-B) \longrightarrow \mathcal{O}_{A+B} \longrightarrow \mathcal{O}_B \longrightarrow 0,$$

where A, B are effective divisors on X. From this we get another exact sequence

$$H^1(\mathcal{O}_C(-kC)) \longrightarrow H^1(\mathcal{O}_{(k+1)C}) \longrightarrow H^1(\mathcal{O}_{kC}) \longrightarrow 0, \tag{$\star$}$$

and so, if $H^1(\mathcal{O}_{k_0C}) = 0$ for some $k_0 \geq 1$, then $H^1(\mathcal{O}_{kC}) = 0$ for all $1 \leq k \leq k_0$. Now assume that y is a rational singularity, or equivalently that $R^1\pi_*\mathcal{O}_X = 0$. Since in particular $R^1\pi_*\mathcal{O}_U$ vanishes for any neighbourhood U of C, by Remark 1.3.11 we have $h^1(\mathcal{O}_{kC}) = 0$ for $k \gg 0$, and the first part of the proof gives the implication. Conversely, there exist arbitrarily small neighbourhoods U of C such that $H^1(U, \mathcal{O}_U) = 0$, and so $R^1\pi_*\mathcal{O}_X|_U = R^1\pi_*\mathcal{O}_U = 0$ for all these neighbourhoods. Therefore $R^1\pi_*\mathcal{O}_X = 0$. □

We can refine our criterion to identify rational singularities with the next theorem, occasionally called *Artin's criterion.* Recall that the *arithmetic genus* of a projective variety Z is defined as $p_a(Z) = (-1)^{\dim Z}(\chi(\mathcal{O}_Z) - 1)$. If Z is a *connected* curve, then $p_a(Z) = g(Z)$ is the usual arithmetic genus of the genus formula.

Theorem 1.3.13. *The contraction of a connected curve $C = \bigcup C_i$ produces a rational singularity if and only if each effective divisor $D \in \langle C_i \rangle_{\mathbb{Z}}$ has arithmetic genus $p_a(D) \leq 0$.*

Proof. One direction is clear: if the contraction produces a rational singularity, we have $h^1(\mathcal{O}_{kC}) = 0$ for every $k \geq 1$. But if $D = \sum k_iC_i$ and $k = \max k_i$, then we have a surjection $r\colon \mathcal{O}_{kC} \to \mathcal{O}_D$, and the long exact sequence in cohomology gives

$$0 = H^1(\mathcal{O}_{kC}) \longrightarrow H^1(\mathcal{O}_D) \longrightarrow H^2(\mathrm{Ker}(r)) = 0$$

since D, kC have dimension 1; therefore $p_a(D) = 1 - h^0(\mathcal{O}_D) + h^1(\mathcal{O}_D) = 1 - h^0(\mathcal{O}_D) \leq 0$.
Conversely, assume that $p_a(D) \leq 0$ for every effective combination D of the components C_i. Considering $D = C_i$, we have

$$0 \geq p_a(C_i) = 1 - h^0(\mathcal{O}_{C_i}) + h^1(\mathcal{O}_{C_i}) = h^1(\mathcal{O}_{C_i}) \geq 0,$$

hence each C_i is smooth and rational. This is the base step for an inductive argument on $k = \sum k_i$ to show that $h^1(\mathcal{O}_D) = 0$ (this is clearly sufficient to prove the implication), so take $D = \sum k_iC_i$ and put $D_i = D - C_i$ for a certain index i appearing in D. By the inductive hypothesis we have $h^1(\mathcal{O}_{D_i}) = 0$, and so it is natural to consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{C_i}(-D_i) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0.$$

By looking at the long exact sequence in cohomology we see that it is sufficient to show that $h^1(\mathcal{O}_{C_i}(-D_i)) = 0$, or equivalently that $C_iD_i \leq 1$, since C_i is rational. If by contradiction $C_iD_i \geq 2$ for every index i, we have $DC_i \geq 2 + C_i^2$ for every i, hence

$$\deg(K_D) = D^2 + K_XD = \sum k_i(DC_i + K_XC_i) = \sum k_i(DC_i - 2 - C_i^2) \geq 0,$$

and $p_a(D) = 1 + \frac{1}{2}\deg(K_D) > 0$, a contradiction. □

Corollary 1.3.14. *A-D-E curves contract to rational singularities.*

Proof. Let $C = \bigcup C_i$ be an *A-D-E* curve. Then $K_XC_i = -2 - C_i^2 = 0$, and for any effective divisor $D = \sum k_iC_i$ we have

$$\deg(K_D) = D^2 + K_XD = D^2 \leq -2$$

from the negative definiteness of the intersection matrix (and since the number $D^2 + K_XD$ is always even). Therefore $p_a(D) = 1 + \frac{1}{2}\deg(K_D) \leq 0$, as desired. □

Again, a simple application of Theorem 1.3.10 specifies two other nice features of *A*-*D*-*E* curves:

Proposition 1.3.15. *Every A-D-E curve C has arbitrarily small neighbourhoods U with $K_U = \mathcal{O}_U$.*

Proof. By Remark 1.3.11 , $H^1(U, \mathcal{O}_U) = H^2(U, \mathcal{O}_U) = 0$ for sufficiently small neighbourhoods U of C, and so $\operatorname{Pic}(U) = H^2(U, \mathbb{Z})$. If we choose U such that C is a deformation retract of U, we have that

$$\operatorname{Pic}(U) = H^2(U, \mathbb{Z}) = H^2(C, \mathbb{Z}) = \bigoplus \mathbb{Z}C_i,$$

hence K_U can be written as a sum of the components C_i. But $K_X C_i = 0$ for each index i, thus K_U is trivial. □

Let U be a neighbourhood of C as above. Every effective divisor $D \in \operatorname{Div}(U)$ can be written in the form $D = D_C + D'$, where $D_C \in \bigoplus \mathbb{Z}C_i$ and every component of D' intersects C in a finite number of points. Notice that there is no well defined intersection form on the open variety U; however, if $D = D_C + D'$ is a divisor as above, we can define the intersection DC_i as the sum $D_C C_i + D'C_i$: the first term is well defined on C, while $D'C_i$ is just the number of intersections (with multiplicities) of D' and C_i.

Proposition 1.3.16. *An effective divisor $D \in \operatorname{Div}(U)$ is linearly equivalent to 0 (i.e., there exists an holomorphic function f on U with $(f) = D$, and we shall write $D \sim 0$) if and only if $D_C C_i \leq 0$ for all i.*

Proof. First of all, notice that an effective divisor $D \in \operatorname{Div}(U)$ is linearly equivalent to 0 if and only if its class in $\operatorname{Pic}(U)$ is 0, and so if and only if $DC_i = 0$ for every i (reasoning as above). Now let $D \sim 0$; since $D'C_i \geq 0$ for every i, necessarily $D_C C_i \leq 0$ for every i. Conversely, if $D_C C_i \leq 0$ for every i, we can choose irreducible curves U_j intersecting C in finitely many points such that $C_i(D_C + \sum U_j) = 0$ for every i: take the U_j's to intersect transversally only one component, and take as many U_j's intersecting C_i as $-D_C C_i$, for each i. □

Now let $D, D' \in \operatorname{Div}(U)$ be effective divisors such that $D \sim D' \sim 0$, and choose f, g holomorphic functions such that $D = (f)$ and $D' = (g)$. For almost all $\alpha, \beta \in \mathbb{C}$, the holomorphic function $\alpha f + \beta g$ has divisor D'' such that $D''_C = \min\{D_C, D'_C\}$; thus there exists a minimal effective divisor Z of the form D_C for some effective divisor $D \in \operatorname{Div}(U)$ such that $ZC_i \leq 0$ for all i. Classically, this divisor is called the *fundamental cycle* of the singularity.

Our work on Dynkin diagrams lets us identify immediately the fundamental cycle of every *A*-*D*-*E* curve. Indeed, embed every Dynkin diagram into the corresponding extended Dynkin diagram; the only vertex F in the difference has multiplicity 1. Then choose Z such that $Z + F$ is the generator of the kernel of the associated form. Clearly $ZC_i = -FC_i \leq 0$ for all i, and if Z weren't minimal, there would exist an index j such that again $(Z - C_j)C_i \leq 0$ for all i. But this is impossible, because

$$(Z - C_j)C_j = (Z + F)C_j - (F + C_j)C_j \geq 1.$$

Therefore Z is the desired fundamental cycle. Since by inspection $ZF = -2$ in any case, we have $0 = (Z + F)^2 = Z^2 + 4 - 2$, hence $Z^2 = -2$.

Another peculiarity of *A*-*D*-*E* curves we will exploit is the following:

Proposition 1.3.17. *Let C be an A-D-E curve, and denote by $y \in Y$ the point obtained from the contraction of C. Then locally around y Y is the double covering of a smooth surface.*

Proof. Let U be a neighbourhood of C as above; let U' be its image in Y. We claim that it is sufficient to find two holomorphic functions f, g defined on U such that:

1. $C = \{f = g = 0\}$;

2. $(f) = \sum k_i C_i + R$, where R is a smooth curve intersecting C either in one point, at which $g|_R$ vanishes to the second order, or two points, at which $g|_R$ vanishes to the first order.

Indeed, if we had these two functions, we could consider the proper map $\varphi\colon U' \to \mathbb{C}^2$ given by $\varphi = (f, g)$ (f, g pass to the quotient): then $\varphi^{-1}(0,0) = y$, and the second property would imply that the degree of φ is 2 in a sufficiently small neighbourhood of $(0,0)$. Therefore, possibly after shrinking U', the map φ just defined would exhibit Y locally as a double covering of a small neighbourhood of $0 \in \mathbb{C}^2$.
Now recall that y is a rational singularity, therefore by Proposition 1.3.16 it suffices to find two distinct effective divisors D_1, D_2 as above such that $D_1C_i, D_2C_i \leq 0$ for all i. But this is easy: if the fundamental cycle Z contains a component C_j of multiplicity 2 such that $ZC_j = -1$, just consider $D_1 = Z + R$ and $D_2 = Z + S$, where R, S are two distinct curves intersecting C_j transversally in one point and no other component C_i. Notice that such a component exists for all cases, except the A_n's. In this case, denote by C_1 and C_n the first and the last curve of the graph, and consider 4 distinct curves R_1, R_2, S_1, S_2 such that R_1, S_1 intersect only C_1 transversally in one point, and R_2, S_2 intersect only C_n transversally in one point. Then it is immediate to see that $D_1 = Z + R_1 + R_2$ and $D_2 = Z + S_1 + S_2$ work. □

Motivated by this result, we want to study the singularities of double coverings, in order to understand the role of *A*-*D*-*E* curves among them. We begin with the central definition of this section:

Definition 1.3.18. The *simple surface singularities* (or *du Val singularities*) are exactly the singularities of double coverings branched over a curve with an *A*-*D*-*E* singularity. We will call the simple surface singularities with the name of the corresponding singularity in the branched curve.

Our earlier work on *A*-*D*-*E* singularities gives us the following classification of simple surface singularities:

Name	**Normal form**
A_n, $n \geq 1$	$w^2 + x^2 + y^{n+1} = 0$
D_n, $n \geq 4$	$w^2 + y(x^2 + y^{n-2}) = 0$
E_6	$w^2 + x^3 + y^4 = 0$
E_7	$w^2 + x(x^2 + y^3) = 0$
E_8	$w^2 + x^3 + y^5 = 0$

Table 1.5: List of possible simple surface singularities. The double covering is given by the projection on the x, y variables.

Clearly, we would like to understand the singularity arising over the singular point $y = (0,0)$ of the branch curve $B \subseteq Y$. The standard method to obtain a resolution of the singularities of $(0,0,0) \in X$ is given by the *canonical resolution* (see [CF99] for a complete discussion), which we now explain. Fix a double covering $\pi : X \to Y$, branched along the curve $B \subseteq Y$, and let $q \in B$ be a singular point for B. Denote by $\mu = 2k + \mu_2$ the multiplicity of B at q, with $\mu_2 \in \{0, 1\}$. Now we blow up q with $\epsilon\colon Y_1 \to Y$ and, if E is the exceptional divisor over q, we obtain a double covering $\pi_1\colon X \times_Y Y_1 \to Y_1$ branched along the (possibly non-reduced) curve $\epsilon^* B = \widetilde{B} + \mu E$, where $\widetilde{B}$ is the strict transform of B in Y_1. Therefore, if $\mu \geq 2$, the domain $X \times_Y Y_1$ is not normal; we claim that its normalization is the double covering X_1 of Y_1 branched along $\widetilde{B} + \mu_2 E$. For, let the space X be defined by the equation $z^2 = f(x, y)$; then, in local coordinates (u, v) of Y_1, where $x = u$ and $y = uv$, we have that $f(x, y) = f(u, uv) = u^\mu g(u, v)$, where $g(u, v)$ defines the strict transform $\widetilde{B}$ of B, and it is not divisible by u. The pullback $X \times_Y Y_1$ is defined by the equation $z^2 = u^\mu g(u, v)$, and it is not normal if $\mu \geq 2$. However, we can consider the new variable $w = \frac{z}{u^k}$ and the new (normal) double covering X_1 of Y_1 given by the equation $w^2 = u^{\mu_2} g(u, v)$ and branched along the curve $B_1 = \widetilde{B} + \mu_2 E$; it is immediate

to check that X_1 is the normalization of $X \times_Y Y_1$.
Working similarly on B_1, we obtain $B_2, \dots, B_m$, until B_{m+1} is smooth near q: this is possible because Proposition 1.3.2 says that at a certain point our reduced total transform B_m has at worst nodes, hence B_{m+1} is precisely the strict transform of B_m and it is smooth. If B has more singular points, we can just apply this resolution simultaneously to all of them. It is easy to notice that in general this does not produce a minimal resolution; however it is well-known that a minimal resolution is obtained by contracting all the (-1)-curves appearing after the canonical resolution.

We can apply this method to resolve simple surface singularities. Let X be the surface given by $w^2 + x^2 + y^{n+1} = 0$; when we blow up the point $(x, y) = (0, 0)$, we put $x = uv$, $y = u$ and we get a new equation (for the normalized double covering)

$$w^2 + v^2 + u^{n-1} = 0.$$

Over the exceptional divisor $\{u = 0\}$ we have produced two distinct lines (spanned by the vectors $(w, v, u) = (1, \pm i, 0)$) intersecting at the point $(0, 0, 0)$ with a singularity of type A_{n-2}. Repeating this argument, we see that the canonical resolution of an A_n simple surface singularity is an *A*-*D*-*E* curve of type A_n.
Consider now the D_n-surface given by $w^2 + y(x^2 + y^{n-2}) = 0$; first, we work in the chart $x = uv$, $y = u$. Then the normalized double covering is given by

$$w^2 + u(v^2 + u^{n-2}) = 0,$$

hence over the exceptional divisor $\{u = 0\}$ we have the double line spanned by $(w, v, u) = (1, 0, 0)$. Similarly, working in the other chart $x = u$, $y = uv$, we obtain the equation

$$w^2 + uv(1 + u^{n-4}v^{n-2}) = 0,$$

and again over the exceptional divisor $\{u = 0\}$ we have a double line. By this explicit description, we have that we have produced two incident double lines, and one of them has a point with a singularity of type D_{n-2}. Using a simple inductive argument, we only have to resolve the D_4 and D_5 surfaces. The first one is fairly simple: as we have seen in Theorem 1.3.4, the normalized double covering contains the curve $uv(v^2 + 1)$, which is a D_4 curve (it has 3 nodes). For D_5, reasoning as above we see that the normalized double covering contains two double lines, of which one contains an A_3 singularity (see Theorem 1.3.4); we resolve it as shown in the figure below.

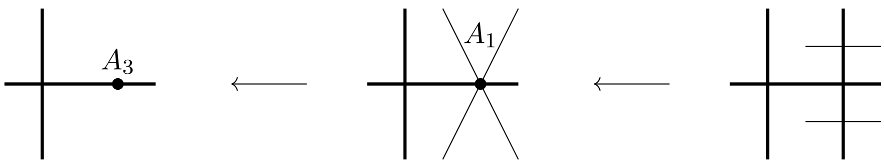


Figure 1.2: The D_5 singularity is resolved by an *A*-*D*-*E* curve of type D_5. The thick line indicates a double line.

The recursive argument we have given above shows that a singularity of type D_n is resolved by a D_n curve.

The E_6 singularity is given by the equation $w^2 + x^3 + y^4 = 0$. Blowing up the point $(x, y) = (0, 0)$ of the branch curve given by $x^3 + y^4 = 0$ we obtain a new equation

$$w^2 + u(v^3 + u) = 0$$

for the normalized double covering. Over the exceptional divisor $\{u = 0\}$ we have a double line F_1. Blow up again the origin of the branch curve and, in the new coordinates $u = u_1 v_1$, $v = u_1$, the double covering now is defined by

$$w^2 + v_1(u_1^2 + v_1) = 0.$$

This produces two lines F_2, F_3 (spanned by $(w, v_1, u_1) = (1, \pm i, 0)$) intersecting F_1 at their intersection point $(w, v_1, u_1) = (0, 0, 0)$. Blow up again the origin of the branch curve, put $v_1 = u_2 v_2$, $u_1 = u_2$, and obtain the new equation for the double covering

$$w^2 + v_2(u_2 + v_2) = 0.$$

Let F_4 be the exceptional divisor $\{u_2 = 0\}$ (notice that F_4 is a triple line, since it is the exceptional divisor corresponding to a triple point). In this coordinates F_2 and F_3 are given by the equations $\{w \pm i v_2 = 0\}$, and since the surface $\{w^2 + v_2(u_2 + v_2) = 0\}$ has two A_1 singularities inside $F_4 = \{u_2 = 0\}$ precisely at the intersections $F_2 \cap F_4$, $F_3 \cap F_4$, we conclude the resolution blowing up these two points and obtaining an E_6 curve. See the figure below to have a graphic interpretation of the resolution.

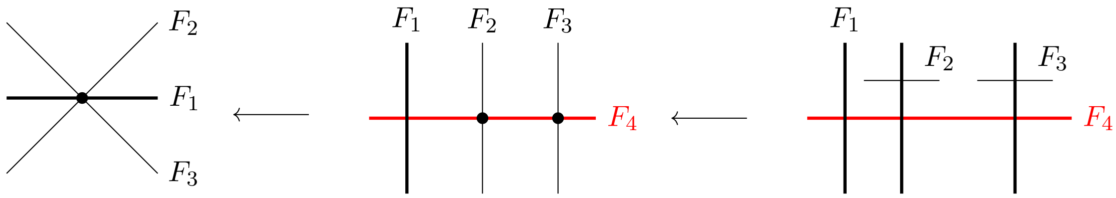


Figure 1.3: The E_6 singularity is resolved by an A-D-E curve of type E_6. The red line indicates a triple line.

We can deal similarly with the E_7 and E_8 singularities, and complete the proof of the following assertion:

Theorem 1.3.19. *A simple surface singularity is resolved by an A-D-E curve of the corresponding type.*

The simple surface singularities arise naturally in many concrete situations. For instance, we will use the following:

Proposition 1.3.20. *Let the group $G = \mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}^2$ as*

$$\overline{1} \cdot (u, v) = (\zeta_{\frac{n}{k}} u, \zeta_n^{-1} v),$$

where $\zeta_n = \exp(\frac{2\pi i}{n})$ and $k \mid n$. Then the image of $(0,0)$ in the quotient $S = \mathbb{C}^2/G$ is a singularity of type $A_{\frac{n}{k}-1}$.

Proof. We will only sketch the proof. By [BPV84, Proposition III.5.3], we have that the singularity is of type $A_{\frac{n}{k},\frac{n}{k}-1}$, i.e. it is isomorphic to the singularity of the (open) surface

$$W = \{(w, u, v) \in \mathbb{C}^3 \mid z^{\frac{n}{k}} = uv\},$$

that is clearly an $A_{\frac{n}{k}-1}$ singularity. $\square$

The final aim of our study is to prove the converse of Theorem 1.3.19: in this way we would obtain a complete characterization of simple surface singularities. In the following, X will denote a normal surface, and Y a smooth surface. First, let us recall a simple general result (see [BPV84, Section I.17] for a proof):

Lemma 1.3.21. *Let $f \colon X \to Y$ be the n-cyclic covering ramified along a smooth divisor B and determined by a line bundle L such that $L^n = \mathcal{O}_Y(B)$. Then*

$$K_X = f^*(K_Y \otimes L^{n-1}) \qquad \textit{and} \qquad f_*\mathcal{O}_X = \bigoplus_{i=0}^{n-1} L^{-i}.$$

Theorem 1.3.22. *Let $f\colon X \to Y$ be a double covering, with branch curve B. Let L be the line bundle inducing f, i.e. such that $L^2 = \mathcal{O}_Y(B)$. Consider the canonical resolution*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\sigma"] \arrow[d, "\widetilde{f}"'] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "\epsilon"'] & Y \end{tikzcd}$$

Then there exists a divisor $Z \geq 0$ on $\widetilde{X}$, with support contained in the union of the exceptional curves for σ, such that

$$K_{\widetilde{X}} = (f \circ \sigma)^*(K_Y \otimes L)(-Z),$$

and $Z = 0$ if and only if the singularities of X are simple.

Proof. Let's work with one blow-up at a time: write $\epsilon = \epsilon_1 \circ \epsilon'$, with $\epsilon_1\colon Y_1 \to Y$ the first blow-up of a point $y \in Y$.

$$\begin{tikzcd} X_1 \arrow[dr] \arrow[drr, bend left, "\sigma_1"] \arrow[ddr, bend right, "f_1"'] & & \\ & X \times_Y Y_1 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & Y_1 \arrow[r, "\epsilon_1"'] & Y \end{tikzcd}$$

The double covering $X \times_Y Y_1 \to X$ is branched along the curve $\widetilde{B} + \mu_y E_y$, where E_y is the exceptional divisor over y and μ_y is the multiplicity of $y \in B$. The normalization of the double covering yields a double covering $f_1\colon X_1 \to Y_1$ branched along $B_1 = \widetilde{B} + (\mu_y)_2 E_y$, where $(\mu_y)_2 \in \{0, 1\}$, hence the line bundle L_1 giving this covering is

$$L_1 = (\epsilon_1^* L)\left(-\left[\frac{\mu_y}{2}\right] E_y\right),$$

since

$$L_1^2 = \mathcal{O}_{Y_1}(B_1) \otimes \mathcal{O}_{Y_1}((\mu_y - (\mu_y)_2)E_y) \otimes \mathcal{O}_{Y_1}\left(-\left[\frac{\mu_y}{2}\right] E_y\right)^2 = \mathcal{O}_{Y_1}(B_1).$$

Therefore, since $K_{Y_1} = \epsilon_1^* K_Y(E_y)$, we have

$$K_{Y_1} \otimes L_1 = \epsilon_1^*(K_Y \otimes L)\left(\left(1 - \left[\frac{\mu_y}{2}\right]\right) E_y\right).$$

Repeating this argument for the remaining blow-ups in ϵ', we get the equality

$$K_{\widetilde{Y}} \otimes \widetilde{L} = \epsilon^*(K_Y \otimes L)(\widetilde{Z}),$$

where $\widetilde{L}^2 = \mathcal{O}_{\widetilde{Y}}\left(\widetilde{B} + \sum_y (\mu_y)_2 E_y\right)$ is the line bundle on $\widetilde{Y}$ inducing the double covering $\widetilde{f}$, and $-\widetilde{Z}$ is a positive divisor (since the multiplicities μ_y are at least 2) with support contained in the union of the exceptional curves for ϵ. Hence we can put $Z = \widetilde{f}^* \widetilde{Z}$ and conclude the first part of the proof by Lemma 1.3.21. It only remains to check the last statement, concerning the simple singularities.
Clearly $Z = 0$ if and only if $\mu_y \leq 3$ for all y, i.e. there are only double and triple points on B and on all of its reduced total transforms. Since *A-D-E* singularities are exactly those with this property, we are done recalling that simple surface singularities are the singularities of double coverings branched over curves with *A-D-E* singularities. □

This theorem shows why simple surface singularities are said to *not affect adjunction*: the canonical resolution $\sigma\colon \widetilde{X} \to X$ of the singularities of X is such that $K_{\widetilde{X}} = \sigma^* K_X$, and since there are no (-1)-curves in the resolution (see again [CF99] for a simple proof), the canonical resolution is in fact the minimal resolution.

Theorem 1.3.23. *The contraction of an A-D-E curve produces a simple surface singularity with the corresponding name.*

Proof. Let $x \in X$ be the singularity arising from the contraction of the A-D-E curve C. Proposition 1.3.17 gives (at least locally) a double covering $f\colon X \to Y$. Then we just combine the previous theorem with Proposition 1.3.15. $\square$

Simple surface singularities are rational by Corollary 1.3.14, hence we could call them *rational double points*, or *RDP*s. We conclude the section with an immediate consequence of our study.

Theorem 1.3.24. *Let $f\colon X \to Y$ be a double covering branched over a curve B. Then X is smooth if and only if B is smooth, and has only RDPs if and only if B has only points with multiplicity ≤ 3, and no triple tacnodes.*

1.4 Lattice Theory

In this section we want to outline some basic properties and results about lattices that we'll need at the end of the exposition to classify the possible configurations of singular fibers for particular elliptic surfaces. A great reference for a thorough discussion is [Nik79].

A *lattice* $(L, \langle,\rangle)$ is a finitely generated free $\mathbb{Z}$-module L endowed with an *integral* bilinear form $\langle,\rangle$. We will be only interested in lattices with symmetric bilinear forms, called *euclidean* lattices. We will often drop the long notation $(L, \langle,\rangle)$, and we will simply write L for the lattice if there is no ambiguity on the bilinear form.

Fix a $\mathbb{Z}$-basis of L, say $\{e_1, \ldots, e_n\}$. The determinant $d(L)$ of the matrix $(\langle e_i, e_j\rangle)_{i,j}$ is independent of the choice of a basis, thus we can call it the *discriminant* of the lattice. We will say that the lattice L is *non-degenerate* if $d(L) \neq 0$, and *unimodular* if $d(L) = \pm 1$.

As it is standard in linear algebra, we can consider the *dual* $L^* = \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, that is again a finitely generated free $\mathbb{Z}$-module. If we denote by $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ the extension of the scalars of L to $\mathbb{Q}$, the bilinear form $\langle,\rangle$ over L naturally extends to a $\mathbb{Q}$-valued bilinear form on $L_{\mathbb{Q}}$; if we start with a non-degenerate symmetric form over L, it will extend to a non-degenerate symmetric form on $L_{\mathbb{Q}}$. In the following, we will assume these two properties for the form $\langle,\rangle$. Define

$$L^{\#} = \{x \in L_{\mathbb{Q}} \mid \langle x, l\rangle \in \mathbb{Z} \ \forall l \in L\}.$$

Clearly $L \subseteq L_{\mathbb{Q}}$ (we are identifying $L \subseteq L \otimes_{\mathbb{Z}} \mathbb{Q}$ with the submodule $L \otimes 1$), since the form $\langle,\rangle$ is $\mathbb{Z}$-valued on L. Moreover, the natural homomorphism

$$\begin{array}{rccc} \phi\colon & L^{\#} & \longrightarrow & L^* \\ & x & \longmapsto & \langle x, \bullet\rangle \end{array}$$

is an isomorphism: if $M = (\langle e_i, e_j\rangle)_{i,j}$ is the matrix associated with the bilinear form, then the functional $\phi(x)$ is such that $\phi(x)(y) = \langle x, y\rangle = {}^t x M y$, thus the inverse $\phi^{-1}(f)$ of a functional $f \in L^*$ is given (in coordinates with respect with the basis $\{e_1, \ldots, e_n\}$) by x, where

$${}^t x = M^{-1} \begin{pmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{pmatrix}.$$

Consequently, $L^{\#}$ is a finitely generated free $\mathbb{Z}$-module, with same rank as L; therefore L has finite index in $L^{\#}$, and we can consider the quotient $G_L = L^{\#}/L$, called the *discriminant form group*. The order of the finite abelian group G_L is $|d(L)| = |\det(M)|$: this is obvious if M is diagonal, and in general the Smith normal form assures that there exist two matrices $N_1, N_2 \in \mathrm{GL}(n, \mathbb{Z})$ (thus with determinant ± 1) such that $N_1 M N_2$ is diagonal. Moreover, G_L is generated by the cosets of the generators of $L^{\#}$, hence

$$\operatorname{length}(G_L) \leq \operatorname{rk}(L^{\#}) = \operatorname{rk}(L),$$

where the *length* of a finite abelian group is the minimum number of generators.

From now on we will consider only *even* lattices, i.e. lattices with even bilinear forms. In other words, the number $\langle l, l\rangle$ must be even for every $l \in L$. Consider the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form q_L on $L^{\#}$ given by

$$q_L(x, x) = \frac{1}{2}\langle x, x\rangle \pmod{\mathbb{Z}}.$$

Since $\langle,\rangle$ is even on L, we see that the quadratic form q_L descends to a well-defined quadratic form (that we will again denote q_L) on G_L. The bilinearity of $\langle,\rangle$ gives

$$q_L(nx) = n^2 q_L(x)$$

for every $x \in G_L$ and $n \in \mathbb{Z}$, while the usual polarization formula shows that

$$\langle x, y\rangle_{G_L} = q_L(x+y) - q_L(x) - q_L(y)$$

for every $x, y \in G_L$, where the bilinear form $\langle , \rangle_{G_L}$ is the induced $\mathbb{Q}/\mathbb{Z}$-valued bilinear form on G_L.

We are finally ready to show some examples. The ones we will be mostly interested in are the lattices A_n, D_n, E_6, E_7, E_8 corresponding to the Dynkin diagrams: explicitly, L is the free $\mathbb{Z}$-module generated by the vertices of the graph, and $\langle , \rangle$ is the associated bilinear form defined at the beginning of Section 1.2. From the definition, these lattices are symmetric, non-degenerate, and even. In the following proposition, we are going to compute the group G_L for each lattice L coming from a Dynkin diagram.

Proposition 1.4.1. *Let L be an A_n, D_n, E_6, E_7 or E_8 lattice. Then the group G_L is isomorphic to one of the following groups:*

L	G_L
A_n	$\mathbb{Z}/(n+1)\mathbb{Z}$
D_n, n even	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
D_n, n odd	$\mathbb{Z}/4\mathbb{Z}$
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8	$\{0\}$

Table 1.6: Discriminant form groups for lattices corresponding to Dynkin diagrams.

Proof. If L is one of E_6, E_7 or E_8, then it suffices to compute $d(L)$. An easy computation of the determinants of the corresponding matrices yields that $d(L) = 3, 2, 1$ respectively, and there exists only one group with order respectively $3, 2, 1$.
Consider $L = A_n$, and let $v_1, \dots, v_n$ be the vertices, i.e. a $\mathbb{Z}$-basis for L. The bilinear form is given by

$$v_i v_j = \begin{cases} -2 & \text{if } i = j \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if $x = \sum_{i=1}^n q_i v_i \in L_\mathbb{Q}$, we have that $x \in L^\#$ if and only if all the numbers

$$\langle \sum_{i=1}^n q_i v_i, v_j \rangle = \begin{cases} -2q_1 + q_2 & \text{if } j = 1 \\ -2q_n + q_{n-1} & \text{if } j = n \\ -2q_j + q_{j-1} + q_{j+1} & \text{if } j \neq 1, n \end{cases}$$

are 0 in $\mathbb{Q}/\mathbb{Z}$. Now $q_2 = 2q_1$ in $\mathbb{Q}/\mathbb{Z}$, and from the third equation we get

$$-2q_2 + q_1 + q_3 = 0,$$

i.e. $q_3 = 3q_1$ in $\mathbb{Q}/\mathbb{Z}$. Applying this recursively, we get $q_j = jq_1$ in $\mathbb{Q}/\mathbb{Z}$ for every j. It only remains to impose the second condition

$$-2nq_1 + (n-1)q_1 = 0,$$

i.e. $(n+1)q_1 = 0$. Therefore each $q_1 = \frac{i}{n+1}$, with $1 \leq i \leq n+1$, uniquely determines a coset in $L^\#/L$, which is a cyclic group of order $n+1$ generated by the element

$$\frac{1}{n+1} \sum_{i=1}^n i v_i.$$

Now consider $L = D_n$. As above, we need that all the following numbers are 0 in $\mathbb{Q}/\mathbb{Z}$:

$$\langle \sum_{i=1}^{n} q_i v_i, v_j \rangle = \begin{cases} -2q_1 + q_2 & \text{if } j = 1 \\ -2q_j + q_{j-1} + q_{j+1} & \text{if } 2 \leq j \leq n-3 \\ -2q_{n-2} + q_{n-3} + q_{n-1} + q_n & \text{if } j = n-2 \\ -2q_j + q_{n-2} & \text{if } j = n-1, n \end{cases}$$

where we are labelling with $n-2$ the only degree 3 vertex, and with $n-1$ and n the two outer vertices. Reasoning as above, we get $q_j = jq_1$ for every $1 \leq j \leq n-2$ (as numbers in $\mathbb{Q}/\mathbb{Z}$). Now we impose the other conditions

$$\begin{cases} 2q_{n-1} = (n-2)q_1 \\ 2q_n = (n-2)q_1 \\ 2(n-2)q_1 = q_{n-1} + q_n + (n-3)q_1 \end{cases}, \text{ from which } \quad \begin{cases} 2(q_{n-1} + q_n) = 2(n-2)q_1 \\ q_{n-1} + q_n = (n-1)q_1 \end{cases}$$

From the last two equalities we get $2q_1 = 0$ in $\mathbb{Q}/\mathbb{Z}$. Now, if n is even, we have $2q_n = 0$, $2q_{n-1} = 0$ and $q_{n-1} + q_n = q_1$, so we have two choices for q_{n-1} (0 or $\frac{1}{2}$), the same two choices for q_n, and q_{n-1}, q_n uniquely determine q_1. In the end, G_L is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by the two elements with $(q_{n-1}, q_n) = (0, \frac{1}{2})$ and $(q_{n-1}, q_n) = (\frac{1}{2}, 0)$.
If instead n is odd, we have $q_{n-1} + q_n = 0$ and $2q_n = (n-2)q_1 = q_1$. Therefore $4q_n = 0$, and the 4 choices for $q_n = \frac{i}{4}$, with $i = 0, 1, 2, 3$, uniquely determine q_1 and q_{n-1}. We conclude that G_L is isomorphic to the cyclic group $\mathbb{Z}/4\mathbb{Z}$. □

In particular we have shown that the lattice E_8 is a rank 8, negative-definite, even, unimodular lattice. Actually, the converse is also true:

Theorem 1.4.2. *The lattice E_8 is, up to isometry, the only rank 8, negative-definite, even unimodular lattice.*

The proof of this theorem is rather lengthy and technical, and goes beyond the scope of this brief introduction. For a detailed proof, we refer to [Ser73].

Thanks to the previous explicit computation, we get some interesting consequences. For instance, if $L = A_n$ and we identify G_L with $\mathbb{Z}/(n+1)\mathbb{Z}$ (say that x is the generator for G_L found in the proof above), then

$$\begin{aligned} q_L(x) &= \frac{1}{2}\langle x, x \rangle = \frac{1}{2(n+1)^2} \sum_{i,j} ij \langle v_i, v_j \rangle = \\ &= \frac{1}{2(n+1)^2} \left[-2\sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n-1} i(i+1) \right] = \frac{1}{(n+1)^2} \left[-n^2 + \frac{n(n-1)}{2} \right] = \\ &= \frac{n}{2(n+1)^2}(-n-1) = \frac{-n}{2(n+1)}. \end{aligned}$$

This has an immediate consequence:

Proposition 1.4.3. *If the lattice L is A_4, A_3, $A_2 \oplus A_2$ or $A_1 \oplus A_1 \oplus A_1$, then G_L has no non-zero isotropic elements.*

Proof. First, we notice that none of the A_i, with $1 \leq i \leq 4$, contains non-zero isotropic elements. Now, if $(a, b) = (\alpha x, \beta y) \in A_2 \oplus A_2$ is isotropic, where x, y are the generators as above, then

$$-\frac{1}{3}(\alpha^2 + \beta^2) \in \mathbb{Z},$$

i.e. $\alpha^2+\beta^2=0 \pmod 3$. It is immediate to notice that necessarily $\alpha=\beta=0 \pmod 3$. For the last lattice $L=A_1\oplus A_1\oplus A_1$ we proceed similarly: we search for triples $(a,b,c)\in\{0,1\}^3$ such that

$$-\frac{1}{4}(a^2+b^2+c^2)\in\mathbb{Z},$$

and we immediately realize that necessarily $a,b,c=0$. □

Remark 1.4.4. If L is a lattice of the form $L=L_{i_1}\oplus\ldots\oplus L_{i_k}$, where L_{i_j} is a finite sum of A_{i_j} lattices and $i_1+1,\ldots,i_k+1$ are pairwise coprime, and G_L contains non-trivial isotropic elements, then at least one $G_{L_{i_j}}$ contains non-trivial isotropic elements. This is straightforward: first of all notice that there can exist at most one odd i_j, and without loss of generality we can assume that it is i_1. Now, if $a_1,\ldots,a_k$ are elements such that $a_j\in G_{L_{i_j}}$, then

$$\sum_{j=1}^{k} q_{L_{i_j}}(a_j)\equiv 0 \pmod{\mathbb{Z}}$$

implies (multiplying by the odd number $P=\prod_{j=2}^{k}(i_j+1)^2)$ that $q_{L_{i_1}}(a_1)\equiv 0 \pmod{\mathbb{Z}}$, since P and $2(i_1+1)$ are coprime. Then $a_1=0$ or it corresponds to a non-trivial isotropic element in $G_{L_{i_1}}$, and we can iterate the argument.

The study of the isotropic elements is particularly important, as highlighted by the following result:

Proposition 1.4.5. *There exists a* 1-1 *correspondence*

$$\left\{\begin{matrix}\textit{intermediate lattices } L\subseteq M\subseteq L^{\#}\\ \textit{with } \langle,\rangle|_M \ \mathbb{Z}\textit{-valued and even}\end{matrix}\right\}\longleftrightarrow\left\{\begin{matrix}q_L\textit{-isotropic}\\ \textit{subgroups } H<G_L\end{matrix}\right\}$$

Moreover, if the intermediate lattice M corresponds to the isotropic subgroup H, then $G_M\cong H^{\perp}/H$, and q_M is induced from q_L.

Proof. Take an intermediate even lattice M, and consider the quotient $H=M/L$. Since $\langle,\rangle|_M$ is even, clearly $q_L(M/L)=0$ by definition. Conversely, take a q_L-isotropic subgroup $H<G_L$, and consider the obvious projection $\pi\colon L^{\#}\to G_L$. Then $M=\pi^{-1}(H)$ is an intermediate lattice; moreover the form $\langle m,n\rangle|_M=q_L(m+n)-q_L(m)-q_L(n)$ must be $\mathbb{Z}$-valued, and it is even since

$$\langle m,m\rangle|_M=q_L(2m)-2q_L(m)=4q_L(m)-2q_L(m)=2q_L(m).$$

To prove the last assertion, we have only to show that, if M corresponds to H, then $\pi^{-1}(H^{\perp})=M^{\#}$. But this is clear: $x\in H^{\perp}$ if and only if $\langle x,y\rangle=0 \pmod{\mathbb{Z}}$ for every $y\in H=M/L$, if and only if $\langle\pi^{-1}(x),m\rangle\in\mathbb{Z}$ for every $m\in M$, i.e. $\pi^{-1}(x)\in M^{\#}$. □

We conclude the section with the last result we will need.

Remark 1.4.6. Let $L\subseteq L'$ be lattices of the same rank n, and let $[L':L]$ be the index of L in L'. Choose $\mathbb{Z}$-bases $\{e_1,\ldots,e_n\}$ of L' and $\{f_1,\ldots,f_n\}$ of L, and let A be the integral matrix such that $f_i=Ae_i$ for every i. Clearly $[L':L]=|\det(A)|$, and moreover

$$(\langle f_i,f_j\rangle)_{i,j}={}^tA(\langle e_i,e_j\rangle)_{i,j}A,$$

thus $d(L)=[L':L]^2d(L')$.

Theorem 1.4.7. *Let U be a unimodular even lattice, and L_1, L_2 two non-degenerate sublattices of U, such that $L_1=L_2^{\perp}$ and $L_2=L_1^{\perp}$. Then there exists an isomorphism $G_{L_1}\cong G_{L_2}$ that carries the form q_{L_1} into the form $-q_{L_2}$.*

Proof. By our assumptions, $L = L_1 \oplus L_2$ is a non-degenerate sublattice of U; since the form of U is $\mathbb{Z}$-valued, clearly U is a sublattice of $L^\#$. Therefore, our previous correspondence says that there exists an isotropic subgroup $H < G_L$ corresponding to U. Notice that the group G_L splits as $G_{L_1} \oplus G_{L_2}$, by our assumptions $L_1 = L_2^\perp$ and $L_2 = L_1^\perp$; denote by π_i the projection $G_{L_1} \oplus G_{L_2} \to G_{L_i}$, $i = 1, 2$.
Consider the restriction $\pi_1|_H \colon H \to G_{L_1}$; we want to prove that this map is actually an isomorphism. First, let $\pi_1(h) = 0$ for some $h \in H$: by definition of π_1, $h = (0, g_2)$ for some $g_2 \in G_{L_2}$, i.e. h is the coset of $u = (0, x_2) \in U$, for some $x_2 \in L_2^\#$. Clearly $u \in L_1^\perp$, because $\langle (0, x_2), (x_1, 0)\rangle = \langle 0, x_2\rangle + \langle x_1, 0\rangle = 0$, thus $u \in L_2$, i.e. $x_2 \in L_2$, and therefore $g_2 = 0$ in the quotient G_{L_2}. The surjectivity can be proved by looking at the cardinalities: we have just proved that $|H| \leq |G_{L_i}|$ for $i = 1, 2$ (the case $i = 2$ is identical); moreover, it is true that $|H|^2 = |G_L| = |G_{L_1}| \cdot |G_{L_2}|$. For, recall that $|H| = [U : L]$ and $|G_L| = |d(L)|$, thus $d(L) = [U : L]^2 d(U) = [U : L]^2$ by the previous remark. Now the surjectivity is immediate combining the equality $|H|^2 = |G_{L_1}| \cdot |G_{L_2}|$ with the inequalities $|H| \leq |G_{L_i}|$, $i = 1, 2$.
To complete the proof of the assertion, we have to provide the isomorphism between G_{L_1} and G_{L_2}. Obviously, we put $f = (\pi_2|_H) \circ (\pi_1|_H)^{-1} \colon G_{L_1} \to G_{L_2}$; it is an isomorphism by construction, and if $h \in H$, then

$$0 = q_L(h) = q_{L_1}(\pi_1(h)) + q_{L_2}(\pi_2(h)),$$

i.e. $q_{L_1} = -q_{L_2}$ under f. □

Corollary 1.4.8. *Let U be a unimodular lattice, and $L \subseteq U$ a non-degenerate sublattice of U. Then $G_{L^\perp} \cong G_{L^{\perp\perp}}$ and $q_{L^\perp} = -q_{L^{\perp\perp}}$.*

Proof. Since $L^{\perp\perp\perp} = L^\perp$, we can just apply the previous theorem with $L_1 = L^\perp$ and $L_2 = L^{\perp\perp}$. □

Chapter 2

Elliptic Surfaces

This chapter is devoted to the study of the basic properties of Elliptic Surfaces. Our aim is to introduce the most important objects and tools that we are going to need in the last part of the thesis. There are many great references about Elliptic Surfaces, such as the classical [Kod60], [Kod63a], [Kod63b] by Kodaira; however we will follow [Mir89], which presents the topic from a more modern viewpoint. Further references will be given in specific points during the exposition. From now on, we will work over the field $\mathbb{C}$ of complex numbers.

2.1 First Definitions and Examples

Definition 2.1.1. Let C be a smooth curve. An *elliptic surface* over C is a smooth surface X with a holomorphic map $\pi\colon X \to C$ whose general fiber is a smooth connected curve of genus 1.

Observe that we haven't said that the general fiber is an elliptic curve, since we haven't fixed an origin. If we were to choose a point on it, this would imply that in every fiber there is a given origin, and so we would get a section of the fibration π. An elliptic surface is a *Jacobian surface* if it admits a section; although we will be mainly interested in Jacobian surfaces, we want to underline that these form a special subset of elliptic surfaces. Good references for general elliptic surfaces are [BPV84] and [FM94].

We will often indicate the elliptic surface (X, π) simply by X, when there is no ambiguity on the fibration π. Moreover, if a section $s\colon C \to X$ of π is given, we will identify the section s with its image $S = s(C) \subseteq X$.

Definition 2.1.2. Let (X, π) be an elliptic surface. We say that a curve on X is *vertical* if it is contained in a fiber of π; otherwise we say that it is *horizontal.*

Definition 2.1.3. An elliptic surface X is said *minimal elliptic* if it has no vertical (-1)-curves.

Note that a minimal elliptic surface is not necessarily minimal in the classical sense, since it could contain horizontal (-1)-curves; we will specify the adjective *elliptic* to avoid ambiguities.

It is a common philosophy in Algebraic Geometry to understand the behaviour of fibrations under base changes, and elliptic surfaces are no exception: we state in the next lemma the basic properties that we will use throughout the exposition.

Lemma 2.1.4. *Let $\pi\colon X \to C$ be an elliptic surface, and let $f\colon C' \to C$ be a flat morphism of curves. Consider the cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\pi} \\ C' & \xrightarrow[f]{} & C \end{array}$$

where $\pi' \colon X' = X \times_C C' \to C'$ is the pull-back of π under f. Then:

1. *$\pi' \colon X' \to C'$ is an elliptic surface.*

2. *If π has a section s, then s induces a section s' of π'.*

3. *If f is branched only over points of C corresponding to smooth fibers and X is smooth, then X' is smooth.*

4. *With the hypotheses of the previous point, if X is smooth and minimal elliptic, then so is X'.*

Proof. 1. Since the general fiber of π is a smooth elliptic curve and does not belong to the branch locus, Hurwitz's formula implies that also the general fiber of π' is a smooth elliptic curve.

2. We have a diagram

$$\begin{tikzcd}
C' \arrow[dr, dashed, "s'"] \arrow[drr, bend left, "s \circ f"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X' \arrow[r, "f'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\
& C' \arrow[r, "f"'] & C
\end{tikzcd}$$

and from the universal property of the pull-back we obtain a morphism $s' \colon C' \to X'$ such that $\pi' \circ s' = \mathrm{id}$, i.e. s' is a section of π'.

3. f' is locally a covering of X, hence X' has at most singularities over singular points of the branched locus. But the hypotheses assure that the branched locus is smooth.

4. If E' is a vertical (-1)-curve on X', its image $f'(E')$ cannot be a single point, thus $E = f'(E')$ is a rational curve (since the genus can't increase). Therefore E is contained in a singular fiber, hence f' is étale around E. Write $K_{X'} = f^*K_X + R$, with $RE' = 0$ by what we have just said, and let d be the degree of $f'|_{E'} \colon E' \to E$. Since E' is a vertical (-1)-curve, $K_{X'}E' = -1$; however

$$-1 = K_{X'}E' = (f^*K_X)E' + RE' = (f^*K_X)E' = K_X(f_*E') = K_X(dE) = dK_XE,$$

thus $d = 1$ and E is a vertical (-1)-curve on X, a contradiction.

□

Obviously, the smooth fibers of an elliptic surface can degenerate into singular fibers; the first one to classify the possible singular fibers was Kodaira, and we present below the list following his conventions, which are now commonly accepted.

Name	Type of fiber
I_0	smooth elliptic curve
I_1	nodal rational curve
I_2	2 smooth rational curves meeting transversally at two points
I_n, $n \geq 3$	n smooth rational curves meeting in a cycle
I_n^*, $n \geq 0$	$n+5$ smooth rational curves meeting with dual graph $\widetilde{D}_{n+4}$
II	cuspidal rational curve
III	2 smooth rational curves meeting at one point to order 2
IV	3 smooth rational curves meeting at one point
IV^*	7 smooth rational curves meeting with dual graph $\widetilde{E}_6$
III^*	8 smooth rational curves meeting with dual graph $\widetilde{E}_7$
II^*	9 smooth rational curves meeting with dual graph $\widetilde{E}_8$
${}_mI_n$	topologically an I_n, but each curve has multiplicity m

Table 2.1: List of possibile singular fibers of a smooth minimal elliptic surface.

The dual graph of a singular fiber is the graph of the intersections of its irreducible components: we indicate with a point each component, and we connect two of them if the corresponding components do intersect. We draw the singular fibers listed above with their own dual graph in Figure 2.2; the number near each vertex indicates the multiplicity of the corresponding component of the fiber.

Before proving that these are the only singular fibers that can occur on an elliptic surface, we want to examine several explicit examples, to show how these configurations arise and at the same time to underline the beauty of their geometry.

Example 2.1.5. The first examples we want to study come from pencils of plane curves. Indeed, let C_1 be a smooth cubic curve in $\mathbb{P}^2$, and let C_2 be any other cubic. By Bezout's Theorem, the intersection $C_1 \cap C_2$ consists of 9 points, possibly infinitely near, and so the pencil $P = \{\lambda C_1 + \mu C_2\}_{[\lambda,\mu]\in\mathbb{P}^1}$ generated by C_1 and C_2 has exactly 9 base points. The pencil P induces a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, and after blowing up the 9 base points we obtain a morphism $\pi\colon X \to \mathbb{P}^1$ such that the inverse image of a generic point $[\lambda,\mu] \in \mathbb{P}^1$ is the elliptic curve $(\lambda C_1 + \mu C_2)^\sim$, where the tilde indicates the strict transform of the curve. Hence the morphism π exhibits the rational surface X as being elliptic over $\mathbb{P}^1$. The canonical bundle K_X of X is

$$K_X = -3\epsilon^*\ell + \sum_{i=1}^{9} E_i,$$

where $\epsilon\colon X \to \mathbb{P}^2$ is the composition of the 9 blow-ups, ℓ is a line in $\mathbb{P}^2$, and the E_i are the 9 (possibly reducible) exceptional divisors on X; since C_1 is linearly equivalent to 3ℓ on $\mathbb{P}^2$, we get

$$3\epsilon^*\ell = \epsilon^*C_1 = \widetilde{C}_1 + \sum_{i=1}^{9} E_i,$$

and thus $K_X = -\widetilde{C}_1$. As a result, if D is any irreducible component of any fiber, we have $K_X D = -\widetilde{C}_1 \cdot D = 0$, and by the genus formula

$$g(D) = 1 + \frac{1}{2}D^2,$$

so if D is contained in a reducible fiber, Zariski's lemma forces $D^2 = -2$ and $g(D) = 0$. If instead F is an irreducible fiber, the same argument shows that $g(F) = 1$ and $F^2 = 0$. Anyway, X must be minimal elliptic.

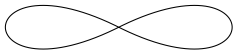
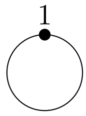
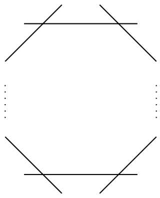
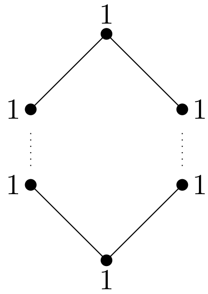
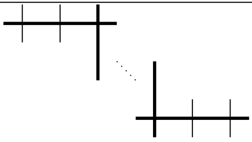
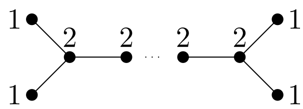
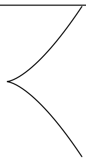
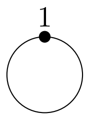
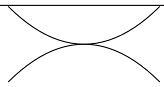
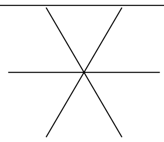
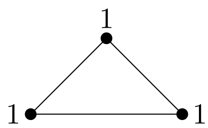
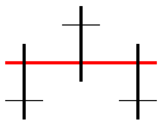
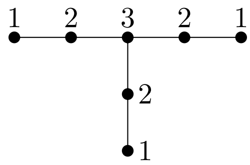
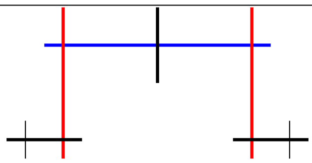
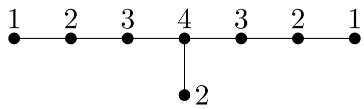
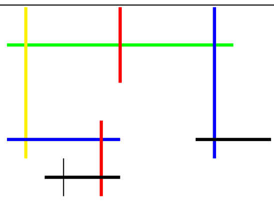
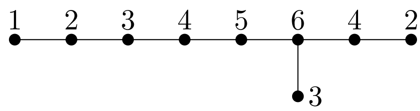
Name	**Type of fiber**	**Dual graph**	**Dynkin diagram**
I_1			$\widetilde{A}_0$
I_n, $n \geq 2$			$\widetilde{A}_{n-1}$
I_n^*, $n \geq 0$			$\widetilde{D}_{n+4}$
II			$\widetilde{A}_0$
III			$\widetilde{A}_1$
IV			$\widetilde{A}_2$
IV^*			$\widetilde{E}_6$
III^*			$\widetilde{E}_7$
II^*			$\widetilde{E}_8$

Table 2.2: Geometric representations and Dynkin diagrams of the possible singular fibers. The last column indicates the name of the dual graph according to the classification of extended Dynkin diagrams discussed in Section 1.2. We are denoting with a black (respectively red, blue, yellow, green) thick line the components with multiplicity 2 (respectively, 3, 4, 5, 6).

Clearly the simplest case is when C_2 is reduced and intersects C_1 transversally in 9 distinct points (away from the singular locus of C_2): the fiber over C_2 is the strict transform of C_2, and thus isomorphic to C_2 itself. Therefore we can obtain the singular fibers of types I_1, II, I_2, III, I_3, IV just choosing as C_2 respectively a nodal curve, a cuspidal curve, a conic plus a line not tangent to the conic, a conic plus a line tangent to the conic, three not concurrent distinct lines and three concurrent distinct lines, as we can see in Table 2.2.

Before passing to analyze the non-reduced case, we ask ourselves the question of how to count singular fibers of such pencils: the answer is given by the next basic lemma, that gives a numerical criterion to determine if a cubic plane curve is singular.

Lemma 2.1.6. *Let $F(x_0, x_1, x_2) = 0$ be a homogeneous cubic equation, denote by H_F the determinant of the Hessian matrix, and put $G_{ij} = x_i \frac{\partial F}{\partial x_j}$ (H_F and the G_{ij} are cubics, too). Let D_F be the determinant of the 10×10 matrix of coefficients of the H_F and the G_{ij}; D_F is homogeneous of degree 12 in the coefficients of F, and the curve C given by $F = 0$ is singular if and only if $D_F = 0$.*

Proof. One implication is clear: if C is singular, the 3 partial derivatives of F share a common solution in $\mathbb{P}^2$, and so do H_F and the G_{ij}, forcing D_F to be 0.
Conversely, let C be smooth; thus we can assume that $F = y^2 - x^3 - Ax - B$ for some constants A, B. A brutal computation of the D_F in this case yields

$$D_F = 2^7 3^3 (4A^3 + 27B^2),$$

i.e. D_F is a multiple of the discriminant Δ for the equation in Weierstrass form. $\square$

This lemma assures that, if we count properly, we get exactly 12 singular fibers for each pencil. Morever, it says that a singular fiber counts as m singular fibers if its equation F gives a zero of order m of D_F. Since the general singular fiber is a nodal curve (of type I_1), we obtain that a general pencil contains exactly 12 singular fibers, each consisting in a nodal curve.

In the next examples we keep the notations given in Example 2.1.5.

Example 2.1.7 (A fiber of type I_0^*). Let C_2 be $2L + M$, where L, M are distint lines in $\mathbb{P}^2$ such that the smooth cubic C_1 intersects L and M transversally in 3 points each.

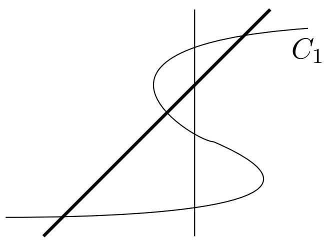


Figure 2.1: A pencil with an I_0^* fiber.

Let p_1, p_2, p_3 be the points of $C_1 \cap L$, and q_1, q_2, q_3 those of $C_1 \cap M$. In a neighbourhood of p_1, the map π is of the form

$$\begin{array}{ccc} \mathbb{A}^2 & \dashrightarrow & \mathbb{P}^1 \\ (x, y) & \longmapsto & [y, x^2] \end{array}$$

where $C_2 = \{x = 0\}$ and $p_1 = (0, 0)$. If we blow up p_1, we get a new domain $\widetilde{\mathbb{A}}^2 = \{(x, y), [u, v] \mid xv = yu\}$. In the chart $u \neq 0$, the exceptional divisor is $E_1 = \{x = 0\}$, and the map becomes

$$(x, y) \longmapsto \left[x\frac{v}{u}, x^2 \right] = \left[\frac{v}{u}, x \right].$$

Since we are interested in the fiber over C_2, we have to look for the inverse image of $[1,0] \in \mathbb{P}^1$, and from the previous explicit form of the map π, we obtain $\pi^{-1}([1,0]) = E_1$ in this chart. Actually this is not quite precise: the indeterminacy is not resolved yet, as the new projection is not defined at $(v,x) = (0,0)$. However it is immediate to notice that the exceptional divisor of the next blow-up cannot belong to the fiber over C_2, and this is because $(v,x) = (0,0)$ is a simple intersection point of two smooth reduced components. We will use this simple remark repeatedly from now on, without mentioning it.
Similarly, in the chart $v \neq 0$ we get a map π of the form

$$(x,y) \longmapsto \left[y, y^2\frac{u^2}{v^2}\right] = \left[1, y\frac{u^2}{v^2}\right],$$

and so $\pi^{-1}([1,0]) = E_1 + 2\widetilde{L}$ in this other chart. Combining the results, we obtain that near p_1, the fiber over C_2 is $2\widetilde{L} + E_1$. Repeating this argument for the other base points, we get a singular fiber over C_2 as shown below, yielding a $\widetilde{D}_4$ dual graph.

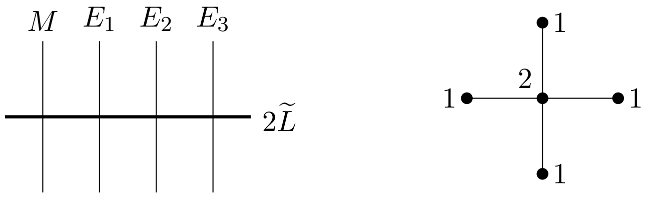


Figure 2.2: An I_0^* fiber and its $\widetilde{D}_4$ dual graph.

Example 2.1.8 (A pencil with constant j). Let C_1 be defined by $y^2z = 0$ and C_2 by $x(x^2 - \alpha xz + z^2) = 0$, with $\alpha \neq \pm 2$. Then the pencil P is given by

$$F = \lambda y^2 z + \mu x(x^2 - \alpha xz + z^2),$$

and an explicit computation shows that $D_F = 2^7 3^3(\alpha^2 - 4)\lambda^6\mu^6$. Thus C_1 and C_2 are the only elements of the pencil with singular fiber; since both λ and μ come with an exponent 6 in D_F, we get that each of C_1 and C_2 contribute 6 times to the counting of all singular fibers.
To understand the types of these singular fibers, we proceed as in Example 2.1.7. The study of the fiber over C_1 is the same as the one done in Example 2.1.7, and for this fiber we get a type I_0^* (in Figure 2.4 we denote $L = \{y = 0\}$, $M = \{z = 0\}$, and the E_i are the exceptional divisors over the points $p_1 = [0,0,1]$, $p_2 = [x_1, 0, 1]$, $p_3 = [x_2, 0, 1]$ of $L \cap C_2$).
Since the exceptional divisors E_i over the 3 points in $L \cap C_2$ are contained in the fiber over C_1, they can't be contained in that over C_2; thus we can restrict to the chart $\{y \neq 0\}$. In this chart the configuration of lines is as below:

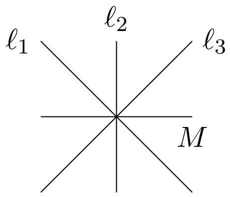


Figure 2.3: The configuration of lines in the chart $\{y \neq 0\}$.

We have to blow up the origin $(0,0)$. Near the origin the map π has the form

$$(x,z) \longmapsto [z, x(x^2 - \alpha xz + z^2)],$$

so after the blow-up this map becomes (in the chart $\{u \neq 0\}$)

$$(x,z),[u,v] \longmapsto \left[x\frac{v}{u}, x^3\left(1-\alpha\frac{v}{u}+\frac{v^2}{u^2}\right)\right] = \left[\frac{v}{u}, x^2\left(1-\alpha\frac{v}{u}+\frac{v^2}{u^2}\right)\right]$$

and so $\pi^{-1}([1,0]) = 2E + \ell_1 + \ell_3$ in this first chart, where E is the exceptional divisor over $(0,0)$. Similarly, in the chart $\{v \neq 0\}$, the map π becomes

$$(x,z),[u,v] \longmapsto \left[z, z^3\frac{u}{v}\left(\frac{u^2}{v^2}-\alpha\frac{u}{v}+1\right)\right] = \left[1, z^2\frac{u}{v}\left(\frac{u^2}{v^2}-\alpha\frac{u}{v}+1\right)\right],$$

and so $\pi^{-1}([1,0]) = 2E+\ell_1+\ell_2+\ell_3$ in this other chart. Now the double line $2E$ intersects transversally ℓ_1, ℓ_2, ℓ_3 and M, and we have to blow up $E \cap M$: it is immediate to see that the exceptional divisor F over $E \cap M$ belongs to the fiber over C_2. Combining the results, we get a diagram of intersections as shown in Figure 2.4, producing another singular fiber of type I_0^*.

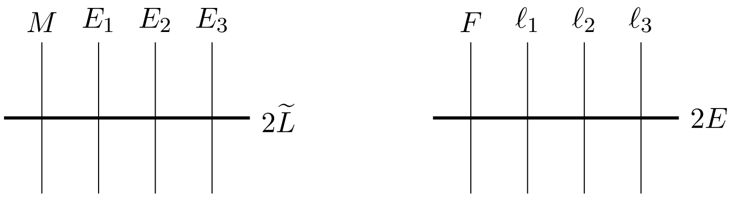


Figure 2.4: The two fibers over C_1 and C_2 are both of type I_0^*.

Since the cubic $\lambda C_1 + \mu C_2$, for $\lambda, \mu \neq 0$, is given by $\{\lambda y^2 + \mu x(x^2 - \alpha x + z^2) = 0\}$, we can see that all these cubics are isomorphic to the smooth plane cubic

$$y^2 = x^3 - \alpha x^2 + x,$$

hence the j-invariant does *not* depend on λ, μ. Thus we have just described a family of isomorphic elliptic curves with 2 degenerations; the fact that we have degenerations assures us that the elliptic surface X is not a product.

Example 2.1.9 (Polygons, or I_n fibers). Let C_1 be as usual a smooth cubic, and let C_2 be the triangle $xyz = 0$. If C_1 passes through a vertex of the triangle, arguments analogous to the ones in the previus examples show that the exceptional divisor over the vertex belongs to the fiber over C_2. Moreover this extra line expands the triangle into a square, since the two sides of the triangle passing through that vertex are disjoint in the blow-up. If in addition C_1 is tangent to one of the sides of the triangle at that vertex, we need to blow up more: the next figures show how to resolve such a base point.

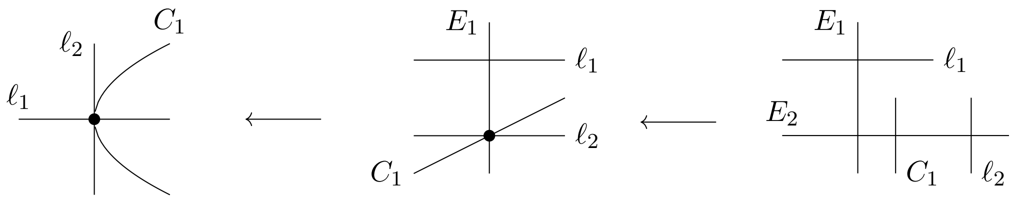


Figure 2.5: Resolution of a base point with consecutive blow-ups. The marked point indicates where we are blowing up.

Since we are considering the fiber over C_2, it is easy to see that both the exceptional divisors E_1, E_2 belong to it, and so the triangle is expanded into a pentagon. As a result, by choosing the right curve C_1 we can expand the triangle into a higher polygon I_n, with $3 \leq n \leq 9$. The extremal case $n = 9$ can be realized by considering the smooth cubic C_1 given by the equation $xy^2 + yz^2 + zx^2 = 0$.

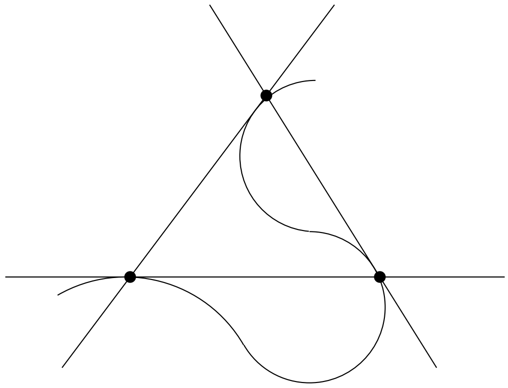


Figure 2.6: A pencil with an I_9 fiber.

Putting $F = \lambda(xy^2 + yz^2 + zx^2) + \mu xyz$, we obtain

$$D_F = 2^3 3^3 \lambda^9 (27\lambda^3 + \mu^3);$$

thus we have a total of 4 singular fibers, of which C_2 counts for 9 and the other 3 for 1. We will see that these last 3 singular fibers are all of type I_1.

Example 2.1.10 (Base change of an I_n fiber). Let $\pi\colon X \to C$ be a smooth minimal elliptic surface with an I_n fiber over the point $p \in C$, and restrict the map π to a fibration $\pi\colon X \to \Delta$, where Δ is a small disc around p. If t is a coordinate on Δ that identifies the point p with the origin, make the base change $t = s^2$. In other words, we need to take the double cover of X branched over the n sides of the n-gon; this produces n A_1 singularities over the nodes of X_0, and after resolving them we obtain a singular fiber of type I_{2n}. Later we will see, using less explicit arguments, that a base change of order m expands an I_n fiber into an I_{mn} fiber.

In the next example we are going to introduce a general strategy to bring an elliptic surface with some singular fibers into its smooth minimal form.

Example 2.1.11. Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the double cover branched over 0 and ∞, and base change the elliptic surface of Example 2.1.8 via f. The branch locus is made of the two singular fibers of type described above, and thus we need to normalize the simple singularities arising from the two double lines; after the normalization, we are taking the double cover of X branched over the 8 multiplicity one lines in the two singular fibers. Near each singular fiber, this double cover consists of a smooth elliptic curve (over the double line) with self-intersection -4 and 4 (-1)-curves (over the 4 other components): for, just apply Hurwitz's formula, using the fact that there are 4 branched points (the 4 points of intersection of the double line with the other components), and since

$$(f^*D)^2 = (\deg f) \cdot (-2) = -4$$

for $D \in \{2\widetilde{L}, M, E_1, E_2, E_3\}$, we obtain the desired self-intersections noticing that $f^*(2\widetilde{L})$ is the elliptic curve and $f^*(D) = 2f^{-1}(D)$ for $D \in \{M, E_1, E_2, E_3, E_4\}$. After blowing down the 8 (-1)-curves, we obtain an elliptic surface with smooth isomorphic fibers; it is not difficult to see that this surface is the product of $\mathbb{P}^1$ with one of the fibers, using the classification of surfaces or the results of Section 2.3.

Example 2.1.12 (A fiber of type I_1^*). Let $C_2 = 2L + M$, as in Example 2.1.7, but choose C_1 tangent to L.

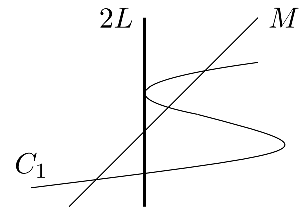


Figure 2.7: A pencil with an I_1^* fiber.

The only singularity not resolved yet is the double point of intersection of C_1 with L; we claim that the resolution transforms the fiber over C_2 as shown below:

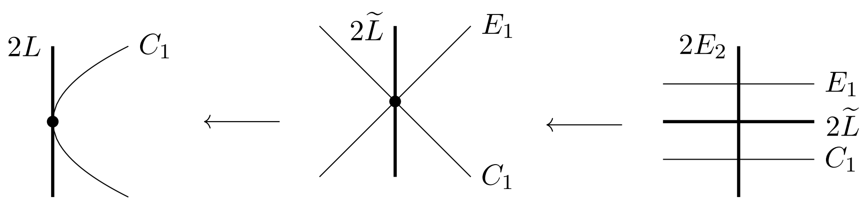


Figure 2.8: Resolution of a quadruple point.

For, let us choose local coordinates near the considered point such that the map π can be assumed to be

$$(x, y) \longmapsto [y - x^2, y^2].$$

After blowing up the origin, in the chart $\{u \neq 0\}$ π becomes

$$(x, y), [u, v] \longmapsto \left[\frac{v}{u} - x, x\frac{v^2}{u^2}\right],$$

where $E_1 = \{x = 0\}$ is the exceptional divisor and $\widetilde{L} = \{v = 0\}$. Since we are interested in the fiber over $[1, 0]$, we need to blow up again the origin. Now π can be written (assume $u = 1$) in the chart $r \neq 0$ as

$$(x, v), [r, s] \longmapsto \left[\frac{s}{r} - 1, x^2\frac{s^2}{r^2}\right],$$

where $E_2 = \{x = 0\}$ is the exceptional divisor and $\widetilde{L} = \{s = 0\}$. Thus, in this chart, the fiber over C_2 is $\pi^{-1}([1, 0]) = 2\widetilde{L} + 2E_2$. Working similarly in the other charts we obtain a diagram of intersections as in Figure 2.8. It remains to blow up the point $E_2 \cap C_1$ and the other point of intersection of C_1 with L; in both cases we get that the exceptional divisor belongs to the fiber over C_2, contributing to a fiber of type I_1^*, as we can see in Figure 2.9.

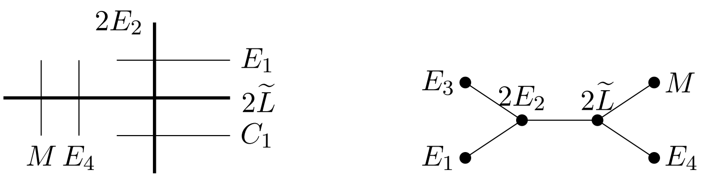


Figure 2.9: An I_1^* fiber and its $\widetilde{D}_5$ dual graph. E_3 is the exceptional divisor over $E_2 \cap C_1$, while E_4 is the exceptional divisor over the third point in $C_1 \cap L$.

Example 2.1.13 (Fibers of types IV^*, III^*, II^*). Let C_2 be a triple line $3L$. We want to show that, varying appropriately C_1, we can obtain a singular fiber of type IV^*, II^* or II^* over C_2. Therefore we consider 3 cases: when C_1 intersects transversally L in 3 distinct points, when C_1 is tangent to L and intersects it in another point, and when L is a flex line for C_1.

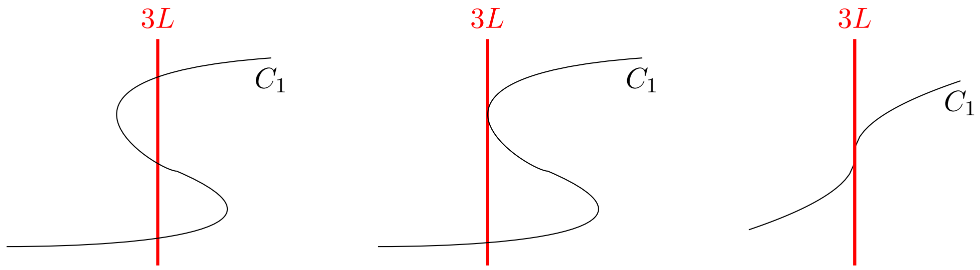
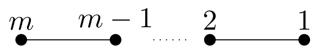


Figure 2.10: The three situation descibed in Example 2.1.13.

Let's examine the first case. In general, if we have a transversal intersection of a curve mL with multiplicity m and a smooth curve C_1, the fiber over mL near the intersection will have a dual graph of the form



as we can see applying $m-1$ times the next argument: if we blow up the intersection point, and we denote by E_1 its exceptional divisor, the fiber over mL becomes

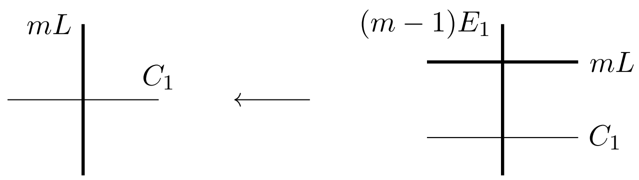


Figure 2.11: Resolution of a point with multiplicity m.

and we go on by blowing up the point $E_1 \cap C_1$. Applying this remark three times in our situation, we see in Figure 2.14 that the fiber over C_2 is of type IV^*.
In the second case, the resolution of the tangency point yields a diagram (near that point) of the form

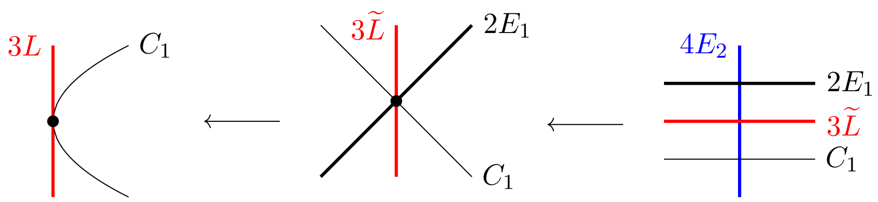


Figure 2.12: Resolution of the tangency point in the fiber over C_2.

It remains to blow up $C_1 \cap E_2$ and the other point in $C_1 \cap L$; the resolutions near those point can be done using Figure 2.11, and at the end we get an III^* fiber over C_2.
Finally, the resolution of the flex point in the third case produces in a neighbourhood a diagram of the form

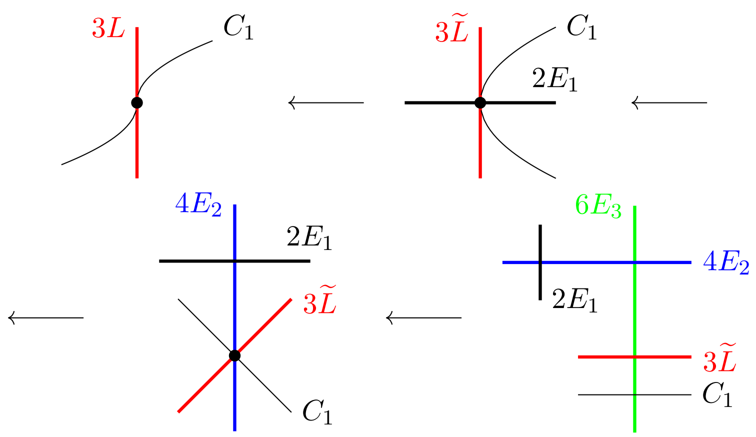


Figure 2.13: Resolution of the flex point in the fiber over C_2.

It remains to blow up $C_1 \cap E_3$, but this can be done using the same remark as before. In conclusion, we get a II^* fiber over the triple line C_2. We draw here below the dual graphs of the singular fiber over C_2 in the 3 cases just examined.

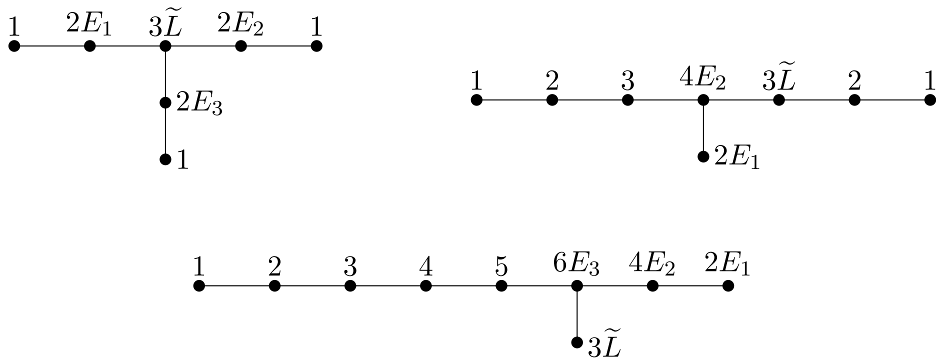


Figure 2.14: Dual graphs (of types IV^*, III^*, II^*) of the singular fiber over C_2 in the 3 described situations.

The last examples are devoted to prove the existence of multiple fibers, although in the discussion we will nearly always consider elliptic surfaces with a section, thus without multiple fibers.

Example 2.1.14 (A fiber of type ${}_mI_0$). Let C be a smooth cubic, and choose 9 points $p_1, \dots, p_9$ on C such that the divisor $\sum_{i=1}^{9} p_i$ does not belong to $|3H|$, but $2\sum_{i=1}^{9} p_i$ belongs to $|6H|$, where H is the hyperplane section of C; such points do exist, as it suffices to choose $p_1, \dots, p_9$ summing (using the sum defined on the elliptic curve C) to a torsion point of order 2. Consider the set of plane sextics double at the points $p_1, \dots, p_9$; since there are $\binom{8}{2} - 1 = 27$ parameters for plane sectics, and imposing a double point gives 3 conditions, necessarily there exists at least one such sextic (not surprisingly, considering that $2C$ has this property). We want to show that there are at least two distinct such sextics: for, take a sextic S double at $p_1, \dots, p_8$, passing through p_9, but having a tangent at p_9 different than that of C. This exists because we are imposing only 26 conditions, 24 for the first 8 points and 2 for the last one. Then there is $p' \in C$ such that

$$2\sum_{i=1}^{8} p_i + p_9 + p' \sim 6H,$$

but $2\sum_{i=1}^{9} p_i \sim 6H$, and so $p' = p_9$, since linearly equivalent points on an elliptic curve are equal. Therefore S must meet C twice at p_9, and since S has a different tangent at p_9 than that of C, p_9 must be a double point for S. Now consider the pencil generated by S and $2C$. A smooth plane sextic has genus $\frac{(6-1)(6-2)}{2} = 10$, but the general member of our pencil has 9 double points, and so its normalization is an elliptic curve. Therefore, after blowing up the base points $p_1, \ldots, p_9$, we obtain an elliptic surface, and we realize that the fiber over $2C$ is exactly $2\widetilde{C}$, and so it is a singular fiber of type ${}_2I_0$.

This example can be easily generalized to plane curves of degree $3m$, in order to obtain a singular fiber of type ${}_mI_0$.

Example 2.1.15 (A double triangle, or a fiber of type ${}_2I_3$). Let Q be a smooth plane conic, and let L_1, L_2, L_3 be 3 distinct lines, tangent to Q.

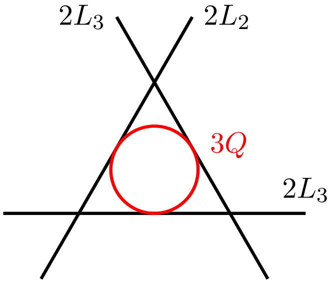


Figure 2.15: The conic Q is the incircle of the triangle $L_1L_2L_3$.

Consider the pencil of sextics generated by $3Q$ and $2(L_1 + L_2 + L_3)$. Resolving the 9 (some of them are infinitely near) base points of the pencil, we notice that a general member C of the pencil has 3 double points over each $Q \cap L_i$, given by the intersections with the 3 exceptional divisors. In other words the strict transform $\widetilde{C}$ of C is linearly equivalent to

$$\widetilde{C} \sim 6H - 2\sum_{i=1}^{3} E_i - 2\sum_{i=1}^{3} F_i - 2\sum_{i=1}^{3} G_i,$$

where H is the hyperplane section and E_i, F_i, G_i are the 9 exceptional divisors. Since after the resolution

$$K_X = -3H + \sum_{i=1}^{3} E_i + \sum_{i=1}^{3} F_i + \sum_{i=1}^{3} G_i,$$

we have

$$\begin{aligned}\widetilde{C}(\widetilde{C} + K_X) &= \left(6H - 2\sum_{i=1}^{3} E_i - 2\sum_{i=1}^{3} F_i - 2\sum_{i=1}^{3} G_i\right)\left(3H - \sum_{i=1}^{3} E_i - \sum_{i=1}^{3} F_i - \sum_{i=1}^{3} G_i\right) = \\ &= 18 - 2\cdot 3 - 2\cdot 3 - 2\cdot 3 = 0,\end{aligned}$$

i.e. $\widetilde{C}$ has genus 1 by the genus formula. Consequently, the resolution of the base points gives us an elliptic surface, and the fiber over the double triangle is simply the double triangle whose sides are the strict transforms of the L_i, contributing to a fiber of type ${}_2I_3$.

This long series of examples proves that the singular fibers listed at the beginning of the section can actually be realized as fibers of elliptic surfaces; we now turn to prove the converse, i.e. these are the only singular fibers that can appear. Our strategy will be subtle: we will use Theorem 1.2.4, saying that the only connected graphs whose associated form is negative semidefinite, with 1-dimensional kernel, are the extended Dynkin diagrams. Clearly, we need to prove that the dual graph of a singular fiber on an elliptic surface has those properties; this will be the task of the next lemmas. Let $\pi\colon X \to C$ be an elliptic surface, with a singular fiber $X_0 = \sum n_i C_i$; the intersection form on X induces a symmetric bilinear form on the $\mathbb{Q}$-vector space $V \subseteq \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the set of components C_i.

Recall the well-known Hodge index theorem (see [BPV84, Corollary IV.2.15]):

Theorem 2.1.16 (Hodge index theorem)**.** *Let D, E be divisors with rational coefficients on the algebraic surface X. If $D^2 > 0$ and $DE = 0$, then $E^2 \leq 0$ and $E^2 = 0$ if and only if E is homologous to* 0*.*

Lemma 2.1.17. *The intersection form on V is negative semidefinite, with a* 1*-dimensional kernel spanned by X_0 itself.*

Proof. Shrinking the base curve C, we can assume that $\pi\colon X \to \Delta$ is a fibration of curves over the unit disc, with a singular fiber X_0 over 0. Since X_0 is linearly equivalent to a generic fiber $X_1 \neq X_0$, $X_0 C_i = 0$ for every i, and so X_0 is in the kernel of the form. Assume now by contradiction the existence of a class $D_1 \in V$ with $D_1^2 > 0$. Clearly $D_1 X_0 = 0$, therefore we can apply the Hodge index theorem to D_1 and $D_2 = X_0$, and we obtain that $D_2 = X_0$ is linearly equivalent to 0 on X, which is absurd since $X_0 > 0$.
It remains to prove that the kernel of the form coincides with $\langle X_0 \rangle$. For, we will the following result: every element in V with self-intersection 0 is a multiple of X_0. By contradiction, let $D \in V$, $D \notin \langle X_0 \rangle$ be such that $D^2 = 0$. Then, there is $\alpha \in \mathbb{Q}$ such that the $\mathbb{Q}$-linear combination $G = D + \alpha X_0$ can be written as $\sum r_i C_i$, with the $r_i \in \mathbb{Q}$ non-zero and not all positive or all negative. Thus $G = P - N$ is a difference of two (non-trivial) effective divisors with distinct components. The product PN is certainly positive, but since X_0 is connected, it is strictly positive; however, $D^2 = 0$ and X_0 is in the kernel of the form, so

$$0 = G^2 = (P - N)^2 = P^2 - 2PN + N^2.$$

But the form is negative semidefinite, so the right hand side is strictly smaller than 0, and we get a contradiction. □

Before proving the classification of the singular fibers, we need to mention a last technicality concerning multiple fibers. Keeping the notation $X_0 = \sum n_i C_i$ for the singular fiber, let $m = \gcd\{n_i\}$; m is said the *multiplicity* of the fiber. Write $X_0 = mF$; we will say that X_0 is *multiple* if $m > 1$.

Lemma 2.1.18. *If $X_0 = mF$ is a multiple fiber, then $\mathcal{O}_X(F)$ and $\mathcal{O}_F(F)$ are torsion line bundles in* $\operatorname{Pic}(X)$*, with order exactly m. In particular, if F is simply connected, then $m = 1$.*

Proof. Shrink the curve C to a small open disc Δ around 0. By the Mittag-Leffler Theorem $\operatorname{Pic}(\Delta) = 0$, and so in particular the line bundle $\mathcal{O}_\Delta(0)$ corresponding to the point $0 \in \Delta$ is trivial. Consequently, its pull-back $\mathcal{O}_X(X_0) = \mathcal{O}_X(mF)$ is trivial, hence $\mathcal{O}_X(F)$ is a torsion line bundle with order dividing m. But if its order were smaller than m, we would have a holomorphic function on X vanishing along X_0 of smaller order than $z \circ f$, z being a coordinate on Δ, which is absurd since every holomorphic function on X is the pull-back of some holomorphic function on Δ.
Now shrink Δ again, so that the restrictions $H^i(X, \mathbb{Z}) \to H^i(F, \mathbb{Z})$ for $i = 1, 2$ are bijections: this is possible, because locally F is a deformation retract of X. The exponential sequence gives a commutative diagram

$$\begin{array}{ccccccc}
H^1(X,\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_X) & \longrightarrow & H^1(\mathcal{O}_X^*) & \longrightarrow & H^2(X,\mathbb{Z}) \\
\| & & \downarrow & & \downarrow & & \| \\
H^1(F,\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_F) & \longrightarrow & H^1(\mathcal{O}_F^*) & \longrightarrow & H^2(F,\mathbb{Z})
\end{array}$$

$\mathcal{O}_F(mF) = \mathcal{O}_F \otimes \mathcal{O}_X(mF)$ is trivial, thus $\mathcal{O}_F(F)$ is a torsion line bundle with order $k \mid m$. Since $H^2(X, \mathbb{Z})$ has no torsion and $\mathcal{O}_X(F) \in H^1(\mathcal{O}_X^*)$ has finite order, there exists a $\xi \in H^1(\mathcal{O}_X)$ mapped into $\mathcal{O}_X(F)$. Moreover $k\xi|_F \in H^1(\mathcal{O}_F)$ is mapped into 0, so there exists a preimage $c \in H^1(F, \mathbb{Z})$ of $k\xi|_F$. Now $\frac{m}{k}c$ and $m\xi$ have the same trivial image in $H^1(\mathcal{O}_F)$, and considering that $H^1(F, \mathbb{Z}) \to H^1(\mathcal{O}_F)$ is injective, necessarily the image of c in $H^1(\mathcal{O}_X)$ is $k\xi$. Therefore $\mathcal{O}_X(kF) = 0$, and so $k = m$.
To prove the last assertion note that, since F is simply connected, we have an exact sequence

$$0 \longrightarrow H^1(\mathcal{O}_F) \longrightarrow \operatorname{Pic}(F) \xrightarrow{c_1} H^2(F, \mathbb{Z}).$$

If $L \in \mathrm{Pic}(F)$ is a torsion line bundle, then it belongs to the kernel of c_1, and so $L \in H^1(\mathcal{O}_F)$, which is torsion-free. $\square$

Theorem 2.1.19. *The only possible singular fibers for a smooth minimal elliptic surface X are those listed in Table 2.1.*

Proof. Write $X_0 = mF$ and $F = \sum r_i C_i$, with the r_i coprime. Assume $m = 1$. The genus of F is 1, so, if F is irreducible, it can be a smooth elliptic curve (type I_0), a rational nodal curve (type I_1) or a cuspidal rational curve (type II). Suppose instead that F is reducible. The adjunction formula on the general fiber X_η gives $K_X X_\eta = 0$, since the canonical bundle on X_η is trivial and $X_\eta^2 = 0$; hence $K_X F = 0$ too, that is

$$0 = \sum r_i C_i \cdot K_X = \sum r_i (2g(C_i) - 2 - C_i^2)$$

by the genus formula. We claim that the numbers $2g(C_i) - 2 - C_i^2$ are all non-negative: for, suppose by contradiction that $2g(C_i) - 2 - C_i^2 < 0$, that is

$$-2 \leq 2g(C_i) - 2 < C_i^2 \leq -1$$

by the negative semidefiniteness of the intersection form over F. The only possibility is $g(C_i) = 0$ and $C_i^2 = -1$, and this is ruled out by minimality. Combining this remark with the previous computation, we obtain that $2g(C_i) - 2 = C_i^2 < 0$, and so $g(C_i) = 0$ and $C_i^2 = -2$ for every i.
Now form the dual graph G to the fiber F, that is, the graph with vertices v_i such that $v_i v_j = C_i C_j$ for every i, j. Lemma 2.1.17 and Theorem 1.2.4 imply that G must be an extended Dynkin diagram different from $\widetilde{A}_0$. If G is $\widetilde{A}_1$, the type of F is I_2 or III, depending on whether the two components intersect in one or two points. If G is $\widetilde{A}_2$, then the type of F is I_3 or III, depending on whether the 3 components meet in a cycle or at a point. If G is one of the other extended Dynkin diagrams, we have no ambiguity in the type of F: if G is $\widetilde{A}_n$, with $n \geq 3$, necessarily F is an I_{n+1} fiber; if G is $\widetilde{D}_n$, with $n \geq 4$, then F is an I_{n-4}^* fiber; if G is $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$, then F is respectively an IV^*, III^* or II^* fiber.
We only have to examine the case of a multiple fiber, i.e. $m > 1$. The previous lemma says that F cannot be simply connected, and so the only possibility is $F = I_n$, for some $n \geq 0$. Then X_0 is a ${}_mI_n$ fiber, and thus we exhaust the list in Table 2.1. $\square$

We conclude the section by proving that every elliptic surface X admits a unique smooth minimal elliptic model; note that this does not imply the uniqueness of a minimal model in the classical sense. Clearly a smooth minimal elliptic model exists, since we can just resolve the singularities on X and then blow down all the vertical (-1)-curves on X.

Definition 2.1.20. Two elliptic surfaces $\pi_1 \colon X_1 \to C$ and $\pi_2 \colon X_2 \to C$ over C are *birational as elliptic surfaces over C* if there exists a birational map $f \colon X_1 \dashrightarrow X_2$ such that the diagram

$$\begin{tikzcd} X_1 \arrow[rr, dashed, "f"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & C & \end{tikzcd}$$

commutes.

Proposition 2.1.21. *Let X_1, X_2 be smooth minimal elliptic surfaces, birational as elliptic surfaces over C via a birational map $f \colon X_1 \dashrightarrow X_2$. Then f is an isomorphism.*

Proof. By the commutativity of the previous diagram, the rational map f can be resolved by blowing up or down points or curves in the fibers of the fibrations π_1, π_2. Consider a minimal diagram

$$\begin{tikzcd} & X \arrow[dl, "\eta"'] \arrow[dr, "\nu"] & \\ X_1 \arrow[rr, dashed, "f"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & C & \end{tikzcd}$$

i.e. such that the number of blow-ups in η is minimal. We claim that ν is an isomorphism; then by simmetry the same would follow for η. Suppose by contradiction that ν contains at least a blow-up, and let E be the first exceptional curve blown down by ν. Clearly E cannot be exceptional for η, otherwise we would deny the minimality of the diagram. Therefore E is the strict transform of a component of a fiber of π_1. E is smooth rational with self intersection -1, so from Theorem 2.1.19 we have that $\eta(E)$ is a nodal or cuspidal rational curve with self-intersection 0: it cannot be smooth elliptic because the genus can't increase, and it cannot be smooth rational since it would have self-intersection -2, and the self-intersection decreases upon blow-ups. Moreover $\eta(E)$ has self-intersection 0. But since the double point of $\eta(E)$ will be blown up at some stage by η, after that we will have

$$0 = (\eta(E))^2 \geq (E + 2E')^2 = E^2 + 4EE' + 2(E')^2 = E^2 + 4$$

where E' is the exceptional divisor at that blow-up, and this is absurd since $E^2 = -1$. □

This immediately implies the uniqueness of the smooth minimal elliptic model:

Corollary 2.1.22. *Given an elliptic surface X over C, there exists a unique smooth minimal elliptic surface X' over C birational to X as elliptic surfaces over C.*

The uniqueness also gives a natural 1-1 corrispondence between the sets

$$\left\{\begin{matrix}\text{smooth minimal elliptic} \\ \text{surfaces over } C\end{matrix}\right\}_{/\cong} \longleftrightarrow \left\{\begin{matrix}\text{curves of genus 1} \\ \text{over } K(C)\end{matrix}\right\}_{/\cong}$$

Indeed, if X is a smooth minimal elliptic surface over C, we can take its generic fiber X_η, which is a curve of genus 1 over $K(C)$; conversely, Corollary 2.1.22 says that there exists a unique smooth minimal elliptic surface over C with the given curve as generic fiber. Clearly, the previous correspondence can be restricted to a 1-1 correspondence between smooth minimal elliptic surfaces over C with section and elliptic curves over $K(C)$.

2.2 Weierstrass Fibrations

The first section was quite general, as we wanted to provide the most general definition for an elliptic surface; however, from now on, we are forced to make some assumptions. This depends on the fact that we want to develop the theory of Weierstrass fibrations, i.e. families of elliptic curves in Weierstrass form: the Weierstrass equation gives an origin on each fiber, hence we must have a section of our fibration $\pi\colon X \to C$, and forces each fiber to be irreducible. The first assumption is quite drastic, and makes our study much less general; anyway, we content ourselves to deal with Jacobian elliptic surfaces, that are (at least) a fundamental special case of elliptic surfaces. On the other hand the latter doesn't change much: if $\pi\colon X \to C$ is a smooth minimal elliptic surface with a section S, for each fiber F we have $SF = 1$, so S must meet exactly one irreducible component of F. This component cannot be multiple, and in particular F itself cannot be multiple. If we denote by $\overline{F}$ the union of the irreducible components of F minus the one meeting S, we can form the dual graph of $\overline{F}$: looking at Table 2.2, we notice that this dual graph is a Dynkin diagram, and it depends only on the fiber type of F. Obviously, if F is irreducible (i.e. it is of types I_0, I_1 or II), the subset $\overline{F}$ is empty; if instead F is reducible, $\overline{F}$ forms a connected set of smooth rational curves with self-intersection -2. For ease of reference, we list below the Dynkin diagram of $\overline{F}$ in every non-trivial case.

Type of fiber	**Dynkin diagram of $\overline{F}$**
I_n, $n \geq 2$	A_{n-1}
I_n^*, $n \geq 0$	D_{n+4}
III	A_1
IV	A_2
IV^*	E_6
III^*	E_7
II^*	E_8

Table 2.3: Dual graphs of $\overline{F}$.

Then, we can contract all these $\overline{F}$, and we get a (possibly singular) elliptic surface with irreducible fibers; however, these are only simple singularities, and we have intensely studied them in Section 1.3. Notice that if the fibration $\pi\colon X \to C$ has a singular fiber of type I_n, $n \geq 1$, over $c \in C$, then, after contracting $\overline{F}$, the new fiber is a nodal rational curve: more precisely, it is immediate to see that the contraction of a single component of a fiber of type I_n, $n \geq 2$, produces a fiber of type I_{n-1} (with a singularity). Similarly, if the fibration has a singular fiber of type II, III or IV, then the contraction of $\overline{F}$ produces a cuspidal rational fiber.

Definition 2.2.1. A *Weierstrass fibration* is a flat and proper morphism $\pi\colon X \to C$ such that every geometric fiber is irreducible of genus 1, the general fiber is smooth, and it is given a section S of π intersecting every fiber in a smooth point.

The previous discussion shows that there is a map

$$F : \left\{ \begin{matrix} \text{smooth minimal elliptic surfaces} \\ \text{over } C \text{ with section } S \end{matrix} \right\}_{/\cong} \longrightarrow \left\{ \begin{matrix} \text{Weierstrass fibrations} \\ \text{over } C \end{matrix} \right\}_{/\cong}.$$

We can also define a "converse" map G that associates to every Weierstrass fibration its unique smooth minimal elliptic model: in other words, it resolves its singularities and blows down the vertical (-1)-curves (compare this to Example 2.1.11). Notice that, by the uniqueness of the smooth minimal elliptic model, the composition $G \circ F$ is the identity; hence F is injective and G is surjective.
Unfortunately, F and G are not inverses: to prove this, we are going to show that G is not injective. Clearly the image of the trivial fibration $X = E \times C$ under G is X itself, since it is smooth and

minimal elliptic, and a section is given by $S = \{O\} \times C$, where O is the origin of E. Now blow up a point $x = (O, c)$: this produces a reducible fiber over c consisting of the (smooth) strict transform of E and the exceptional divisor E' over x, both with self-intersection -1. Blowing down E', we obtain again the product surface; if instead we blow down E, E' becomes a rational cuspidal curve with self-intersection 0, and so the resulting surface X' (that is obviously minimal elliptic over C with section S) is not isomorphic to X, since it has a singular fiber over c. But X' is clearly birational to X as elliptic surfaces, and so, by the uniqueness of the smooth minimal elliptic model, $G(X') = X$.

Definition 2.2.2. A Weierstrass fibration is said to be *in minimal form* if it is in the image of F.

From the previous discussion we have that Weierstrass fibrations over C in minimal form are in 1-1 correspondence with smooth minimal elliptic surfaces over C with section, which in turn are in 1-1 correspondence with elliptic curves over $K(C)$. We will later give a complete characterization of Weierstrass fibrations in minimal form; for the moment, we focus on some of their basic properties.

Let $\pi \colon X \to C$ be a Weierstrass fibration with section S. We have the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(S) \longrightarrow N_{S/X} \longrightarrow 0,$$

where $N_{S/X} = \mathcal{O}_S(S)$ is the normal bundle of S in X. We apply π_* to this exact sequence, and we have another exact sequence

$$0 \longrightarrow \pi_*\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_X(S) \longrightarrow \pi_*N_{S/X} \longrightarrow R^1\pi_*\mathcal{O}_X \longrightarrow R^1\pi_*\mathcal{O}_X(S).$$

π is a proper morphism, so $\pi_*\mathcal{O}_X = \mathcal{O}_C$; moreover, the dimensions $h^i(X_c, \mathcal{O}_{X_c}(nS)) = h^i(X_c, \mathcal{O}_{X_c}(nO))$ for $i = 0, 1$ are constants in c, and so the sheaves $\pi_*\mathcal{O}_X(nS)$ and $R^1\pi_*\mathcal{O}_X(nS)$ are locally free (see [BPV84, Theorem I.8.5]) of ranks $h^0(X_c, \mathcal{O}_{X_c}(nO)) = n$ and $h^1(X_c, \mathcal{O}_{X_c}(nO)) = h^0(X_c, \mathcal{O}_{X_c}(-nO)) = 0$ respectively. Therefore the above exact sequence simplifies in

$$0 \longrightarrow \pi_*\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_X(S) \longrightarrow \pi_*N_{S/X} \longrightarrow R^1\pi_*\mathcal{O}_X \longrightarrow 0.$$

But the same argument as before shows that $R^1\pi_*\mathcal{O}_X$ is a line bundle on C, and so the surjective map $\pi_*N_{S/X} \to R^1\pi_*\mathcal{O}_X$ has to be an isomorphism. This yields the following:

Corollary 2.2.3. *Let $\pi \colon X \to C$ be a Weierstrass fibration with section S. Then*

$$\pi_*N_{S/X} \cong R^1\pi_*\mathcal{O}_X \qquad \text{and} \qquad \pi_*\mathcal{O}_X(S) \cong \pi_*\mathcal{O}_X = \mathcal{O}_C.$$

The line bundle $\mathscr{L} = (\pi_*N_{S/X})^{-1} \cong (R^1\pi_*\mathcal{O}_X)^{-1}$ is called the *fundamental line bundle* on C, and clearly it does not depend on the section S.

Identifying as usual $\mathcal{O}_X((n-1)S)$ with a subsheaf of $\mathcal{O}_X(nS)$ for every n, we get an ascending chain $\pi_*\mathcal{O}_X((n-1)S) \subseteq \pi_*\mathcal{O}_X(nS)$ of sheaves over C; the successive quotients of this chain are line bundles over C, and they are simply powers of the fundamental line bundle $\mathscr{L}$:

Lemma 2.2.4. *For every $n \geq 2$ there is a short exact sequence*

$$0 \longrightarrow \pi_*\mathcal{O}_X((n-1)S) \longrightarrow \pi_*\mathcal{O}_X(nS) \longrightarrow \mathscr{L}^{-n} \longrightarrow 0.$$

Proof. Just apply π_* to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X((n-1)S) \longrightarrow \mathcal{O}_X(nS) \longrightarrow \mathcal{O}_S(nS) \longrightarrow 0$$

and notice that $R^1\pi_*\mathcal{O}_X((n-1)S) = 0$ and $\mathcal{O}_S(nS) = \pi_*N_{S/X}^{\otimes n} = \mathscr{L}^{-n}$, since π is an isomorphism restricted to S. $\square$

Now pick $c \in C$, and consider the sheaf $\pi_*\mathcal{O}_X(3S)$. The stalk over c is $H^0(X_c, \mathcal{O}_{X_c}(3O))$, and Lemma 1.1.6 gives a canonical splitting of this vector space induced by the basis $\{1, x, y\}$, since the Weierstrass basis (x, y) is uniquely determined up to constants. More precisely:

Lemma 2.2.5. *For every $n \geq 2$ there is a splitting*

$$\pi_*\mathcal{O}_X(nS) \cong \mathcal{O}_C \oplus \mathscr{L}^{-2} \oplus \ldots \oplus \mathscr{L}^{-n}.$$

Proof. As before, the basis $\{1, x, \ldots, x^m, y, xy, \ldots, x^{m-2}\}$ of $H^0(X_c, \mathcal{O}_{X_c}(2mO))$ (and a similar one for n odd) induces locally a splitting of the sheaf $\pi_*\mathcal{O}_X(nS)$. By the uniqueness of the Weierstrass basis, these n directions are uniquely determined, and so they give a canonical splitting of $\pi_*\mathcal{O}_X(nS)$. We conclude applying recursively Lemma 2.2.4. □

Therefore we obtain the following situation: chosen a sufficiently fine open cover of C, we have local Weierstrass bases (x, y), and thus the theory of Weierstrass equations locally describes $\pi\colon X \to C$ as a family of elliptic curves; our goal is to patch together these local descriptions.

Let $\{U_i\}$ be a trivializing cover for $\mathscr{L}$ with transition functions $\{\alpha_{ij}\}$: if e_i is a local basis for $\mathscr{L}|_{U_i}$, then we have $e_i = \alpha_{ij} e_j$ on the intersection $U_i \cap U_j$. Lemma 2.2.4 gives us two elements $f_i \in \pi_*\mathcal{O}_X(2S)|_{U_i}$ and $g_i \in \pi_*\mathcal{O}_X(3S)|_{U_i}$ such that f_i projects onto $e_i^{-2} \in \mathscr{L}^{-2}|_{U_i}$ and g_i projects onto $e_i^{-3} \in \mathscr{L}^{-3}|_{U_i}$; imitating the proof of Lemma 1.1.6, we obtain an equation

$$g_i^2 = a_6 f_i^3 + a_5 f_i g_i + a_4 f_i^2 + a_3 g_i + a_2 f_i + a_0$$

as elements in $\pi_*\mathcal{O}_X(6S)|_{U_i}$, where the a_i are regular functions. With our assumptions on f_i and g_i, we have that the left hand side projects onto e_i^{-6}, while the right hand side projects onto $a_6 e_i^{-6}$; so necessarily $a_6 = 1$. Completing the square and the cube in the previous equation as in Lemma 1.1.6, we get a Weierstrass basis (x_i, y_i) such that

$$y_i^2 = x_i^3 + A_i x_i + B_i,$$

where A_i and B_i are local section of $\mathcal{O}_X$. Moreover, since the two processes do not affect the terms of highest order, we have that x_i and y_i transform as e_i^{-2} and e_i^{-3} respectively, i.e.

$$x_i = \alpha_{ij}^{-2} x_j \qquad \text{and} \qquad y_i = \alpha_{ij}^{-3} y_j$$

on the intersection $U_i \cap U_j$. Therefore

$$x_i^3 + A_i x_i + B_i = y_i^2 = \alpha_{ij}^{-6} y_j^2 = \alpha_{ij}^{-6}(x_j^3 + A_j x_j + B_j) = x_i^3 + \alpha_{ij}^{-4} A_j x_i + \alpha_{ij}^{-6} B_j,$$

and so the local sections $A_i e_i^4$ and $B_i e_i^6$ patch together as global sections A, B of $\mathscr{L}^4$ and $\mathscr{L}^6$ respectively.

Definition 2.2.6. The sections (A, B) of $\mathscr{L}^4$ and $\mathscr{L}^6$ are called the *Weierstrass coefficients* for the Weierstrass fibration $\pi\colon X \to C$. The section $\Delta = 4A^3 + 27B^2$ of $\mathscr{L}^{12}$ is called the *discriminant* of the fibration π.

Clearly, the local uniqueness of the sections A_i, B_i implies the uniqueness of the Weierstrass coefficients (A, B) up to the action of $H^0(C, \mathcal{O}_C)^*$ given by $\lambda \cdot (A, B) = (\lambda^4 A, \lambda^6 B)$. Moreover, the discriminant Δ is not identically zero, as its vanishing implies the singularity of the corresponding fiber, and it is well defined up to the action of $H^0(C, \mathcal{O}_C)^*$ given by $\lambda \cdot \Delta = \lambda^{12}\Delta$. Therefore, A and B are not both identically zero.

Let us organize the above construction in a definition:

Definition 2.2.7. A set of *Weierstrass data over* C for the Weierstrass fibration $\pi\colon X \to C$ is a triple $(\mathscr{L}, A, B)$, where $\mathscr{L}$ is a line bundle over C and A, B are global sections of $\mathscr{L}^4$ and $\mathscr{L}^6$ such that the discriminant $\Delta = 4A^3 + 27B^2$ is not identically zero.

Naturally, we say that two triples $(\mathscr{L}_1, A_1, B_1)$ and $(\mathscr{L}_2, A_2, B_2)$ of Weierstrass data over C are isomorphic if there exists an isomorphism $\varphi : \mathscr{L}_1 \to \mathscr{L}_2$ that respects the given sections, i.e. such that A_1 is mapped onto A_2 under φ^4 and B_1 is mapped onto B_2 under φ^6. Then we have another 1-1 correspondence

$$\left\{\begin{matrix}\text{Weierstrass data}\\ \text{over } C\end{matrix}\right\}_{/\cong} \longleftrightarrow \left\{\begin{matrix}\text{Weierstrass fibrations}\\ \text{over } C\end{matrix}\right\}_{/\cong}.$$

Indeed, we have just showed how to derive Weierstrass data from a Weierstrass fibration; moreover, if $(\mathscr{L}, A_1, B_1)$ and $(\mathscr{L}, A_2, B_2)$ are two triples of Weierstrass data for $\pi\colon X \to C$, where $\mathscr{L} = (R^1\pi_*\mathcal{O}_X)^{-1}$, then the uniqueness of the Weierstrass coefficients imply that the two Weierstrass data are isomorphic. Conversely, a triple $(\mathscr{L}, A, B)$ of Weierstrass data over C gives a Weierstrass fibrations over C by patching together the local fibrations given by the surfaces $y_i^2 = x_i^3 + A_i x_i + B_i$.

Another interesting consequence is that the fundamental line bundle has positive degree:

Lemma 2.2.8. *Let* $(\mathscr{L}, A, B)$ *be Weierstrass data over* C*. Then* $\deg(\mathscr{L}) \geq 0$*, and* $\deg(\mathscr{L}) = 0$ *if and only if it is a torsion line bundle in* $\mathrm{Pic}(C)$*, of order* $1, 2, 3, 4$ *or* 6*.*

Proof. The first assertion is immediate, as $\mathscr{L}^{12}$ admits a non-zero section. For the second one, just notice that $\mathscr{L}^4$ (or $\mathscr{L}^6$) is a line bundle with degree 0 and a non-zero section, hence trivial. □

The degree $\deg(\mathscr{L})$ of the fundamental line bundle is extremely important, because the number $12\deg(\mathscr{L})$ equals the degree of the divisor $\{\Delta = 0\}$ on C:

Corollary 2.2.9. *The number of singular fibers of a Weierstrass fibration coincides with the degree* $12\deg(\mathscr{L})$ *of the divisor* $\{\Delta = 0\}$ *on* C*.*

To continue our study of Weierstrass fibrations, we need to dwell on some global aspects: in particular, we have to introduce two alternative representations of Weierstrass fibrations.

The first global representation is motivated by the following remark: if c is a point of C, and $U \subseteq C$ is a neighbourhood of c such that there exists a Weierstrass basis (x, y) for $\pi^{-1}(U)$, then we have a commutative diagram

$$\begin{tikzcd} & \mathbb{P}^2 \times U \arrow[d, "p"] \\ \pi^{-1}(U) \arrow[ur, hook, "{[1,x,y]}"] \arrow[r, "\pi"'] & U \end{tikzcd}$$

where p is the projection, and the map $[1, x, y]$ is injective, since it is so on each fiber of π. We would like to patch these local diagrams to embed X into a $\mathbb{P}^2$-bundle over C, and the theory we have developed allows us do this in a very elegant way.
If $\mathscr{L}$ denotes as usual the fundamental line bundle over C, we have a canonical splitting

$$\pi_*\mathcal{O}_X(3S) \cong \mathcal{O}_C \oplus \mathscr{L}^{-2} \oplus \mathscr{L}^{-3},$$

as $\mathcal{O}_X(3S)$ is locally generated by $\{1, x, y\}$, where (x, y) is a Weierstrass basis for the fibration. Adjunction gives a natural map $\phi\colon \pi^*\pi_*\mathcal{O}_X(3S) \to \mathcal{O}_X(3S)$; ϕ is surjective, because $1, x, y$ locally generate the 3 direct summands of $\pi_*\mathcal{O}_X(3S)$. By [Bea96, III.17] we get a map

$$f\colon X \longhookrightarrow \mathbb{P} = \mathbb{P}(\pi_*\mathcal{O}_X(3S))$$

that coincides locally with the map $[1, x, y]$ considered above: $\mathbb{P}$ is a $\mathbb{P}^2$-bundle over C, and so we have a commutative diagram

$$\begin{tikzcd} & \mathbb{P} \arrow[d, "p"] \\ X \arrow[ur, hook, "f"] \arrow[r, "\pi"'] & C \end{tikzcd}$$

that realizes X as a divisor inside $\mathbb{P}$. In addition, X can be identified with the divisor given by the global equation $y^2z = x^3 + Axz^2 + Bz^3$, where z, x, y, A, B are interpreted as sections of $\mathcal{O}_C, \mathscr{L}^2, \mathscr{L}^3, \mathscr{L}^4, \mathscr{L}^6$ respectively (locally they are well-defined, and we have seen that they transform as sections of those line bundles). We can think of the given section of π as the set of all the origins of the fibers of π, and thus as the set $\{x = z = 0\}$ inside $X \subseteq \mathbb{P}$.

One advantage of this representation is that we can compute the canonical bundle of X using the adjunction formula, as soon as we know the canonical bundle of $\mathbb{P}$. The next general lemma helps us with this:

Lemma 2.2.10. *Let E be a vector bundle of rank r over a smooth projective variety Y, and consider the projective bundle $\pi\colon X = \mathbb{P}E \to Y$ over Y. Then*

$$K_X = \pi^*(K_Y \otimes \det(E))(-r).$$

Proof. Consider the short exact sequences

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \pi^* T_Y \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \pi^* E^\vee \longrightarrow Q \longrightarrow 0; \tag{2}$$

the first one is simply the definition of $T_{X/Y}$, and the second one is the tautological exact sequence over X, so $\mathrm{rk}(Q) = r - 1$. The determinant of E is just its maximal exterior power, so it commutes with the pull-back π^*; hence (2) gives $\pi^* \det(E^\vee) = \mathcal{O}_X(-1) \otimes \det(Q)$. On the other hand, (1) implies that

$$K_X = \det(T_X^\vee) = \det(T_{X/Y}^\vee) \otimes \pi^* \det(T_Y^\vee) = \det(T_{X/Y}^\vee) \otimes \pi^* K_Y.$$

The Euler sequence in this case is

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \otimes \pi^* E^\vee \longrightarrow T_{X/Y} \longrightarrow 0,$$

hence twisting it by $\mathcal{O}_X(-1)$ we get that $T_{X/Y} = Q \otimes \mathcal{O}_X(1)$. Therefore

$$\begin{aligned} \det(T_{X/Y}^\vee) &= \det(Q^\vee) \otimes \mathcal{O}_X(-r+1) = \\ &= \pi^* \det(E) \otimes \mathcal{O}_X(-1) \otimes \mathcal{O}_X(-r+1) = \\ &= \pi^* \det(E) \otimes \mathcal{O}_X(-r) \end{aligned}$$

and so combining the equations we obtain the stated formula. □

Proposition 2.2.11. *The canonical bundle of X is*

$$K_X = p^*(K_C \otimes \mathscr{L})|_X = \pi^*(K_C \otimes \mathscr{L}).$$

In particular $K_X^2 = 0$, and the Kodaira dimension of X is at most 1.

Proof. The divisor $\mathcal{D}$ defining X inside $\mathbb{P}$ has class in $(p^*\mathscr{L}^6)(3)$. By the adjunction formula and Lemma 2.2.10, the canonical bundle of X is

$$K_X = (K_{\mathbb{P}} \otimes \mathcal{D})|_X = [(p^*(K_C \otimes \mathscr{L}^{-5})(-3) \otimes (p^*\mathscr{L}^6)(3)]|_X = p^*(K_C \otimes \mathscr{L})|_X,$$

since $E = \pi_* \mathcal{O}_X(3S) = \mathcal{O}_C \oplus \mathscr{L}^{-2} \oplus \mathscr{L}^{-3}$ in this case. The second assertion follows immediately from the explicit formula: the canonical bundle is pulled back from C, so its self-intersection must be zero. Finally $h^0(nK_X) = h^0(K_C^n \otimes \mathscr{L}^n)$, and Riemann-Roch says that the growth of this number cannot be more than linear. □

This bound on the Kodaira dimension is the best possible: we can easily construct elliptic surfaces with Kodaira dimension $-\infty, 0$ and 1, as the next example shows.

Example 2.2.12. Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic surface arising from a pencil of plane cubics. If we choose as a section of π an exceptional divisor of one of the blow-ups that is not contained in the generic fiber (for instance, we can take the last exceptional divisor), then $\mathscr{L} = \pi_* N_{S/X}^{-1} = \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore Proposition 2.2.11 gives that $K_X = \pi^*\mathcal{O}_{\mathbb{P}^1}(-1) = -F$ equals minus a fiber of π. Clearly the Kodaira dimension in this case is $-\infty$.
Now let again $C = \mathbb{P}^1$, but choose $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1}(2)$. With the same argument as before, we see that $K_X = 0$, and the Kodaira dimension of X is 0. Notice that, thanks to the results in Section 2.3, we see that X is a K3 surface.
Finally, if $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1}(3)$, the resulting elliptic surface must have Kodaira dimension 1.

Another simple consequence is the following:

Proposition 2.2.13. *X is the product of C with the generic fiber if and only if the fundamental bundle $\mathscr{L}$ is trivial. In particular*

$$h^0(C, \mathscr{L}^{-1}) = \begin{cases} 1 & \text{if } X \text{ is a product} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $\mathscr{L}$ is trivial, then $\mathbb{P} = \mathbb{P}^2 \times C$ and A, B are constants, so X is just the product $\{y^2 = x^3 + Ax + B\} \times C \subseteq \mathbb{P}^2 \times C$. Conversely, suppose X is a product $E \times C$. If $S = \{(f(c), c) \mid c \in C\}$ is the given section of the fibration, then the automorphism σ_f of X such that $\sigma_f(e, c) = (e - f(c), c)$ carries S into the section $\{O\} \times C$, and so the bundle $N_{S/X}$ is trivial.
The other statement follows immediately from this and from the fact that $\mathscr{L} \geq 0$. $\square$

Now we can turn to the other global representation of Weierstrass fibrations. In the previous one we considered the vector bundle $\pi_*\mathcal{O}_X(3S)$ over C; it seems reasonable to consider as well the other vector bundle $\pi_*\mathcal{O}_X(2S)$ over C. Since this bundle is isomorphic to $\mathcal{O}_C \oplus \mathscr{L}^{-2}$, the surface $R = \mathbb{P}(\pi_*\mathcal{O}_X(2S))$ is ruled over C, and the projectivization gives a natural map $q\colon R \to C$. Similarly to the above case, the surjective map of vector bundles $\pi^*\pi_*\mathcal{O}_X(2S) \to \mathcal{O}_X(2S)$ induces a commutative diagram

$$\begin{tikzcd} & R \arrow[d, "q"] \\ X \arrow[ur, "g"] \arrow[r, "\pi"'] & C \end{tikzcd}$$

If we restrict the diagram to a certain fiber of π, the map g is simply given by $(x, y) \mapsto x$, where x, y form a Weierstrass basis for the fiber. In other words, extending the Weierstrass basis to a small neighbourhood, g is given by $[1, x, y] \mapsto [1, x]$. Therefore g is a double covering, branched along the curve $T = \{X^3 + AXZ^2 + BZ^3 = 0\}$ and the section $\{Z = 0\}$ of R. We will refer to the curve T as the *trisection* of R; notice that T and the horizontal section $\{Z = 0\}$ are disjoint in R.

This construction of R is quite natural, and imitates the ideas used in the representation of X as a divisor inside $\mathbb{P}$. However, there is a more immediate way to get R: if (x, y) is a local Weierstrass basis for X, then the global involution $\iota\colon X \to X$ such that $\iota(x, y) = (x, -y)$ gives a quotient $X/\langle\iota\rangle$. This quotient is exactly R, since the induced map $X/\langle\iota\rangle \to R$ has degree 1, is finite, and the two surfaces are normal, thus it is an isomorphism.

The direct image of the trivial sheaf $\mathcal{O}_X$ under g can be easily computed:

Corollary 2.2.14. *The direct image of $\mathcal{O}_X$ under the double covering g is given by the formula*

$$g_*\mathcal{O}_X = \mathcal{O}_R \oplus (q^*\mathscr{L}^{-3})(-2).$$

Proof. The trisection T is a divisor on R with class inside $(q^*\mathscr{L}^6)(3)$, while the class of the section $\{Z = 0\}$ is in $\mathcal{O}_R(1)$. Then $(q^*\mathscr{L}^{-3})(-2)$ is the sub-line bundle of $g_*\mathcal{O}_X$ locally generated by y, and we conclude since $\{1, y\}$ is a local basis for $g_*\mathcal{O}_X$. □

The most interesting element of this alternative representation is the trisection T: remember that an elliptic curve $y^2 = x^3 + Ax + B$ is smooth if and only the polynomial on the right hand side has 3 distinct roots; therefore we get:

Corollary 2.2.15. *The fiber X_c of π over $c \in C$ is smooth if and only if the fiber $R_c = q^{-1}(c)$ intersects the trisection T in 3 distinct points.*

We will see below that this correspondence can be extremely strengthened: the local behaviour of the intersection of T with a fiber R_c completely determines the type of singularity occurring in X_c. To get to this fundamental result, we first have to restrict our attention to a class of Weierstrass fibrations we have already defined at the beginning of this section: the Weierstrass fibrations in minimal form, i.e. those in the image of F. Now it is the moment to complete their characterization, using the trisection T.

Proposition 2.2.16. *Let $\pi\colon X \to C$ be a Weierstrass fibration, with Weierstrass data $(\mathscr{L}, A, B)$. Then the following are equivalent:*

1. *π is in minimal form.*
2. *X has only RDPs as singularities.*
3. *The trisection T has no triple tacnodes.*
4. *There is no point $c \in C$ such that $\mu_c(A) \geq 4$ and $\mu_c(B) \geq 6$, where $\mu_c(s)$ indicates the order of vanishing of the section s at c. When this holds, we will say that the Weierstrass data is in minimal form.*

Proof. 1. $\Rightarrow$ 2. is easy, since by Table 2.3 we are contracting an A-D-E curve. Moreover 2. $\Rightarrow$ follows from Theorems 1.3.19 and 1.3.23: the composition $F \circ G$ is the identity on X. 2. $\Leftrightarrow$ 3. follows from Theorem 1.3.24, since the section $\{Z = 0\}$ is smooth and the trisection cannot have points of multiplicity greater than 3. Now we prove the equivalence 3. $\Leftrightarrow$ 4.: if by contradiction there exists $c \in C$ with $\mu_c(A) \geq 4$ and $\mu_c(B) \geq 6$, we can choose a local coordinate t around c such that $t^4 \mid A$ and $t^6 \mid B$. Then we claim that the trisection T has a triple tacnode: the local equation of T is $x^3 + t^4A'(t)x + t^6B'(t) = 0$, hence it has a triple point at $(0, 0)$; if we blow it up, the strict transform has the form

$$x^3 + t^2A'(t)x + t^3B'(t) = 0,$$

and it has again a triple point at $(0, 0)$. Conversely, assume that the trisection $T(x, t) = x^3 + A(t)x + B(t) = 0$ (in local coordinates) has a triple tacnode at $(x_0, 0)$. Therefore the equation $x^3 + A(0)x + B(0) = 0$ has a triple root at x_0, and since there is no term with x^2, necessarily $x_0 = 0$. Moreover, the presence of a triple tacnode at $(0, 0)$ forces the residue class of $T(x, y) \pmod{\mathfrak{m}^4}$ to be x^3, hence $t^2 \mid A(t)$ and $t^3 \mid B(t)$. We blow up $(0, 0)$, and we obtain an equation

$$x^3 + \frac{A(t)}{t^2}x + \frac{B(t)}{t^3} = 0,$$

that again has a triple point at $(0, 0)$, i.e. $t^4 \mid A(t)$ and $t^6 \mid B(t)$, a contradiction. □

This equivalence gives us a concrete way to decide if a Weierstrass fibration is in minimal form: it suffices to check the multiplicities of the two sections A, B. Let $\pi\colon X \to C$ be any Weierstrass fibration and consider the positive divisor $D = \sum_{c \in C} n_c c$, where $n_c = \min\left\{\left[\frac{\mu_c(A)}{4}\right], \left[\frac{\mu_c(B)}{6}\right]\right\}$: in other words, n_c indicates how many times we have to subtract 4 from $\mu_c(A)$ and 6 from $\mu_c(B)$ in order to have

$\mu_c(A) \leq 3$ or $\mu_c(B) \leq 5$. Clearly the sum in D is finite: the previous proposition says that to each c with $n_c > 0$ corresponds a (non simple) singularity on X. Consider a section f of $\mathcal{O}_C(D)$ that vanishes exactly on D; then we have:

Proposition 2.2.17. *Let $\pi\colon X \to C$ be a Weierstrass fibration with Weierstrass data $(\mathscr{L}, A, B)$. The Weierstrass data for the fibration in normal form, i.e. for $F \circ G(X)$, is $\left(\mathscr{L}(-D), \frac{A}{f^4}, \frac{B}{f^6}\right)$.*

Proof. By our assumption on the n_c's, this Weierstrass data is in minimal form. Notice that the new Weierstrass data is well defined: $\frac{A}{f^4}$ is a section of $\mathscr{L}(-D)^4 = \mathscr{L}^4 \otimes \mathcal{O}_C(-D)^4$, and $\frac{B}{f^6}$ is a section of $\mathscr{L}(-D)^6 = \mathscr{L}^6 \otimes \mathcal{O}_C(-D)^6$. We only have to prove that this new Weierstrass fibration is birational to the given one; this can be done by looking at the generic fibers and applying Theorem 1.1.7 with $\lambda = f \in K(C)^*$. $\square$

A simple but fundamental consequence is the following, that identifies birational Weierstrass fibrations:

Proposition 2.2.18. *Two Weierstrass data $(\mathscr{L}_1, A_1, B_1)$ and $(\mathscr{L}_2, A_2, B_2)$ induce birational Weierstrass fibrations if and only if there exists line bundles M_1, M_2 on C and sections f_1, f_2 respectively of M_1, M_2, such that the two Weierstrass data $(\mathscr{L}_1 \otimes M_1, A_1 f_1^4, B_1 f_1^6), (\mathscr{L}_2 \otimes M_2, A_2 f_2^4, B_2 f_2^6)$ are isomorphic.*

Proof. If such line bundles and sections exist, clearly the two Weierstrass fibrations are birational: again look at the general fibers and apply Theorem 1.1.7. Conversely, we put the two triples of Weierstrass data in normal form, and obtain new Weierstrass data $\left(\mathscr{L}_1(-D_1), \frac{A_1}{g_1^4}, \frac{B_1}{g_1^6}\right)$, $\left(\mathscr{L}_2(-D_2), \frac{A_2}{g_2^4}, \frac{B_2}{g_2^6}\right)$. These two Weierstrass data must be isomorphic, since birational Weierstrass fibrations have the same minimal form; therefore we can tensor them by the line bundle $\mathcal{O}_C(D_1 + D_2)$ and the sections $g_1^4 g_2^4$, $g_1^6 g_2^6$, and obtain the desired isomorphism with $M_1 = \mathcal{O}_C(D_2)$, $M_2 = \mathcal{O}_C(D_1)$, $f_1 = g_2$ and $f_2 = g_1$. $\square$

We will denote by $\mathbb{BW}$ the set of Weierstrass data up to birational equivalence. Therefore we obtain a 1-1 correspondence betweeen the set $\mathbb{BW}$ and the set of Weierstrass data in minimal form, considered up to isomorphism.

As we promised, now we can show that the type of the singular fiber over $c \in C$ is completely determined by the intersection of the trisection T with the fiber R_c of the map $q\colon R \to C$. We assume that the Weierstrass fibration $\pi\colon X \to C$ is in minimal form.

Proposition 2.2.19.

1. *If T intersects R_c in 3 distinct points, then X_c is smooth (type I_0).*
2. *If T intersects R_c in $p + 2q$, with $p \neq q$, then q is at worst a double point of T, and:*
   - *If T is smooth at q, then X_c is a nodal rational curve (type I_1).*
   - *If T has a double point at q, of type A_n, then X_c has type I_{n+1}.*
3. *If T intersects R_c in only 1 point p, then T is at worst triple at p, and:*
   - *If T is smooth at p, then X_c is a cuspidal rational curve (type II).*
   - *If T has a double point at p, then:*
     - *If T has a singularity of type A_1 at p, then X_c has type III.*
     - *If T has a singularity of type A_2 at p, then X_c has type IV.*
   - *If T has a triple point at p, then:*
     - *If T has a singularity of type D_n at p, then X_c has type I^*_{n-4}.*
     - *If T has a singularity of type E_6 at p, then X_c has type IV^*.*
     - *If T has a singularity of type E_7 at p, then X_c has type III^*.*

– *If T has a singularity of type E_8 at p, then X_c has type II^*.*

Proof. First of all, notice that, if T intersects R_c only in p, and p is a double point for T of type A_n, then R_c has to be tangent to T in p. Therefore the intersection number of T and R_c at p must be $n+1 \leq 3$, so necessarily $n \leq 2$. Thus the previous list contains all the possibilities. Now recall that X is a double covering of R, and the simple singularity in X_c has the same name of the singularity arising in the intersection between T and R_c; so Table 2.3 helps us decide the type of singular fiber of X_c from the singularity in T. With this in mind, we have only to be careful in the ambiguous cases: I_1 and II (both producing no singularity), I_2 and III (both producing an A_1 singularity), I_3 and IV (both producing an A_2 singularity). The distinction can be done by looking at the number of intersections between T and R_c: in the 3 latter cases (II, III, IV), the fiber X_c of the *minimal* Weierstrass fibration is a cuspidal rational curve, hence T must meet R_c only in one point; instead, in the 3 former cases (I_1, I_2, I_3), the fiber X_c of the *minimal* Weierstrass fibration is a nodal rational curve, hence T must meet R_c in two points. □

The proposition just proved represents an incredibly efficient way to understand the types of singular fibers studying the trisection T; this identification can be made even more concrete using only 3 numerical invariants, that uniquely determine the type of the singular fiber X_c over $c \in C$. We denote $a = \mu_c(A)$, $b = \mu_c(B)$ and $\delta = \mu_c(\Delta)$, where $\Delta = 4A^3 + 27B^2$ is the discriminant (we will say that $\mu_c(f) = \infty$ if f is constantly c in a neighbourhood). Moreover, j is the value of the j-invariant at c, m its multiplicity, χ the topological Euler characteristic of X_c, r the number of components of X_c not meeting the section, d the number of components of X_c with multiplicity 1, RDP the type of simple singularity arising after the contraction of the r components (that coincides with the type of singularity of the trisection T), γ the genus drop contributed by the singularity on T, $\widetilde{T}$ the strict transform of T after a blow-up, and E the exceptional curve. Recall that a point of multiplicity m contributes a genus drop equal to $\frac{m(m-1)}{2}$.

Theorem 2.2.20. *The 3 numbers a, b, δ uniquely determine the type of singular fiber of X_c. In particular:*

Fiber	a	b	δ	j	m	χ	r	d	**RDP**	γ	**Comment**
I_0	0	0	0	?	?	0	0	1	-	0	T intersects
	≥ 1	0	0	0	$3a$	0	0	1	-	0	R_c in 3
	0	≥ 1	0	1	$2b$	0	0	1	-	0	distinct points
I_1	0	0	1	∞	1	1	0	1	-	0	T tangent to R_c
I_n	0	0	n	∞	n	n	$n-1$	n	A_{n-1}	$\left[\frac{n}{2}\right]$	T intersects R_c twice at the double point
I_0^*	2	3	6	?	?	6	4	4	D_4	3	T has an
	≥ 3	3	6	0	$3a-6$	6	4	4	D_4	3	ordinary triple
	2	≥ 4	6	1	$2b-6$	6	4	4	D_4	3	point on R_c
I_n^*	2	3	$n+6$	∞	n	$n+6$	$n+4$	4	D_{n+4}	$3+\left[\frac{n}{2}\right]$	T has a point of type $[3,2]$ on R_c
II	≥ 1	1	2	0	$3a-2$	2	0	1	-	0	T has a flex in R_c at one point
III	1	≥ 2	3	1	$2b-3$	3	1	2	A_1	1	T has a node, with R_c as one tangent

Fiber	a	b	δ	j	m	χ	r	d	**RDP**	γ	**Comment**
IV	≥ 2	2	4	0	$3a-4$	4	2	3	A_2	1	T has a cusp, with R_c as one tangent
IV^*	≥ 3	4	8	0	$3a-8$	8	6	3	E_6	3	$\widetilde{T}$ has a flex in E at one point
III^*	3	≥ 5	9	1	$2b-9$	9	7	2	E_7	4	$\widetilde{T}$ has a node on E, with E as one tangent
II^*	≥ 4	5	10	0	$3a-10$	10	8	1	E_8	4	$\widetilde{T}$ has a cusp on E, with E as tangent

Table 2.4: The a, b, δ table. The ? in the j column means that the j-invariant can be any number $\neq 0, 1, \infty$.

Proof. The r, d colums are clear; so it is the RDP column, as we know from Table 2.3. Recall that $j = \frac{4A^3}{4A^3+27B^2}$; thus the j and m columns are immediate to verify. We focus on the first three columns, i.e. we prove that the types of singular fibers have precisely those a, b, δ. The smooth type is easy: X_c is smooth if and only if $\delta = 0$, hence a, b cannot be > 0 at the same time. Indeed, the first row contains the other 3 possible cases. From now on we will deal with $\delta > 0$.
The I_n fibers, with $n \geq 1$, are precisely the singular fibers such that the Weierstrass fibration has a nodal rational curve, so $a = b = 0$: for, notice that $a = 0$ if and only if $b = 0$, and if $a, b > 0$, then the Weierstrass fibration has a cuspidal rational curve. So let $a = b = 0$; we have to understand which fiber corresponds to every $\delta > 0$. The trisection has a local form near the double point

$$x^3 + (-3 + f(t))x + (2 + g(t)) = 0,$$

for certain functions f, g such that $f(0) = g(0) = 0$. The double point is at $(1, 0)$, and the other intersection with $R_c = \{t = 0\}$ is of course $(-2, 0)$; so, possibly after shrinking the neighbourhood of the double point, we can suppose that T has $\{x = -2\}$ as a (local) component. Therefore we can assume that T has the local form

$$x^2 + (-2 + f_1(t))x + (1 + g_1(t)) = 0$$

near $(1, 0)$, with $f_1(0) = g_1(0) = 0$. The discriminant of this quadratic equation is (up to a constant) $D = f_1^2(t) - 4(f_1(t) + g_1(t))$; if we change (local) variable replacing x with $x + 1 - \frac{f_1}{2}$, we obtain a local equation near the double point

$$x^2 + \left(f_1(t) + g_1(t) - \frac{f_1^2}{4}\right) = x^2 - \frac{D}{4}.$$

But the order of vanishing of D at 0 is δ, because the $\{x = -2\}$ branch does not contribute; therefore the double point is of type $A_{\delta-1}$, so the singular fiber is of type I_δ.
The I_n^* cases are identified by the fact that T has a triple point with singularity D_{n+4}; so the local form for T is

$$x^3 + t^2 f(t)x + t^3 g(t) = 0.$$

The discriminant is $\Delta = t^6(4f^3(t) + 27g^2(t))$, so we can blow up, put $x = tx$ and consider the strict transform

$$x^3 + f(t)x + g(t) = 0.$$

This new equation has a double point at $\{t = 0\}$ by definition of D_n singularities, and its discriminant has order of vanishing equal to $\delta - 6$, so the total singularity is of type $D_{\delta-2}$, and consequently the

singular fiber is of type $I^*_{\delta-6}$. In this way we have also verified the a, b's for the I^*_n case: we just take the a, b's from the I_n case, and we add respectively $2, 3$.
In the remaining cases, we always have that T has a triple point, so $a, b \geq 1$. Let T have the local form

$$x^3 + A(t)x + B(t) = 0.$$

If X_c has type II, then T is smooth at the triple point $(0,0)$, so the residue class of the equation $\pmod{\mathfrak{m}^2}$ must be non-zero. Therefore $b = 1$, and we have no restrictions on a.
If X_c has type III, then T has a node at $(0,0)$, so the residue class of the equation $\pmod{\mathfrak{m}^2}$ must be 0 and the residue class of the equation $\pmod{\mathfrak{m}^3}$ must have two distinct solutions. Therefore $a = 1$ and $b \geq 2$.
If X_c has type IV, then T has a cusp at $(0,0)$, so the residue class of the equation $\pmod{\mathfrak{m}^3}$ must have two coincident solutions. Therefore $b = 2$ and $a \geq 2$.
In the last three cases, T has a triple point at $(0,0)$, so R_c cannot be tangent to T; notice that R_c has local equation $\{t = 0\}$, so the residue class of the equation $\pmod{\mathfrak{m}^4}$ must be x^3, hence $a \geq 3$ and $b \geq 4$.
According to the previous proposition, we only have to see which type of singularity has the strict transform of the equation; therefore we write $f(t) = t^3 f_1(t)$ and $g(t) = t^4 g_1(t)$, and obtain a strict transform

$$x^3 + t f_1(t) x + t g_1(t) = 0.$$

If X_c is of type IV^*, this curve is smooth, therefore $g_1(0) \neq 0$, i.e. $b = 4$.
If X_c is of type III^*, this curve has a node, therefore $f_1(0) = 0$ and $\mu_0(g_1) = 1$, i.e. $a = 3$ and $b \geq 5$.
If X_c is of type II^*, this curve has a cusp, therefore $\mu_0(g_1) = 1$ and $\mu_0(f_1) \geq 1$, i.e. $b = 5$ and $a \geq 4$.
We have to explain the χ and γ columns. For the former, recall the well-known formula the the Euler characteristic

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y);$$

an I_n fiber is the union of a string of $n - 1$ projective lines (having an Euler characteristic equal to $2(n-1) - (n-2) = n$, if we apply repeatedly the above formula) with another projective line intersecting the string in 2 points, thus $\chi(I_n) = n + 2 - 2 = n$. Now the other verifications are immediate; just to give an example, the II^* fiber has an $\widetilde{E}_8$ dual graph, consisting of a string of 8 projective lines (with $\chi = 9$) intersecting in one point another projective line, resulting in a total characteristic $\chi = 9 + 2 - 1 = 10$.
Finally, the genus drop γ can be computed using that a double point has $\gamma = 1$ and a triple point has $\gamma = 3$: for instance, an A_{n-1} singularity has a double point, and its strict transform has a singularity of type A_{n-3}, so it is immediate to prove by induction that $\gamma(I_n) = \left[\frac{n}{2}\right]$, noticing that I_0, I_1 are irreducible and thus we don't have to contract anything. The other cases are quite similar: for instance, a D_{n+4} singularity has a triple point, and its strict transform has a double point of type A_{n-1}, so clearly $\gamma(I^*_n) = 3 + \left[\frac{n}{2}\right]$. The last three cases are immediate, since we have seen that T has a triple point, and its strict transform has respectively no singular points, one node (i.e. a double point), and one cusp (again a double point). □

The work we have done to reach this exhaustive table is immediately rewarded with the next two results, that we can prove just by inspection of the table.

Proposition 2.2.21. *In all cases we have the equality $\chi = \delta$. Moreover, the difference $\chi - r$ is a number between 0 and 2, that classifies the singular fibers in the following way:*

$$\chi - r = \begin{cases} 0 & \text{if } X_c \text{ is smooth} \\ 1 & \text{if } X_c \text{ is of type } I_n, \ n \geq 1 \\ 2 & \text{otherwise} \end{cases}$$

The equality $\chi = \delta$ is surprising, because it relates a geometric invariant as the Euler characteristic with an analitic quantity as the order of vanishing of the discriminant. We will see more about this formula later in the next section.

The proposition naturally divides the singular fibers in two distinct classes: the ones with $\chi - r = 1$ and the others with $\chi - r = 2$. We will say that a singular fiber is *semistable* if its corresponding quantity $\chi - r$ is 1. For the moment the name seems random, but we will understand its meaning during the exposition.

Proposition 2.2.22. *We have the following classification of the fibers with respect to the behaviour of the j-function:*

j	**Fiber**	m
0	I_0, I_0^*	0 (mod 3)
	II, IV^*	1 (mod 3)
	IV, II^*	2 (mod 3)
1	I_0, I_0^*	0 (mod 2)
	III, III^*	1 (mod 2)
∞	I_n, I_n^*, $n \geq 1$	n

Table 2.5: Singular fibers with their j, m invariants. The table has to be read in the following way: if j assumes that value, then the fiber is of those types if and only if m respects that conditition.

In particular we have that, since j can be ∞ only in presence of a singular fiber, the map $j\colon C \to \mathbb{P}^1$, such that $j(c)$ is the j-invariant of the elliptic curve X_c, has a pole of order precisely $n \geq 1$ only in presence of a singular fiber of type I_n or I_n^*. Therefore, if we denote by i_n, i_n^* the number of singular fibers of type respectively I_n, I_n^* (and in general with small letters the number of singular fibers of the corresponding type), we have:

Corollary 2.2.23. $d = \deg(j) = \sum_{n\geq 1} n(i_n + i_n^*)$.

Actually, we can say a lot more about the map j, especially about its ramification. Assume from now on that the map j is not constant. If $j_0 \in \mathbb{P}^1$, denote by $i_0(j_0)$ and $i_0^*(j_0)$ the number of fibers of types respectively I_0 and I_0^* with $j = j_0$; similarly, denote by $k_{j_0}(m)$ the number of points $c \in C$ with $j(c) = j_0$ and $\mu_c(j) = m$. The notations are quite cumbersome, but they can be used to write

$$d = \sum_{m\geq 1} m k_{j_0}(m).$$

Now Table 2.5 gives a lot of information about these $k_{j_0}(m)$:

$$\begin{aligned}
i_0(0) + i_0^*(0) &= \sum_{m\equiv 0\ (3)} k_0(m) \\
ii + iv^* &= \sum_{m\equiv 1\ (3)} k_0(m) \\
iv + ii^* &= \sum_{m\equiv 2\ (3)} k_0(m) \\
i_0(1) + i_0^*(1) &= \sum_{m\equiv 0\ (2)} k_1(m) \\
iii + iii^* &= \sum_{m\equiv 1\ (2)} k_1(m) \\
i_n + i_n^* &= k_\infty(n) \quad \text{for each } n \geq 1
\end{aligned}$$

There are also some easy inequalities concerning these numbers:

Lemma 2.2.24. *1. The number $d-(ii+iv^*)-2(iv+ii^*)$ is non-negative and multiple of 3. Moreover*

$$i_0(0)+i_0^*(0)\leq\frac{1}{3}[d-(ii+iv^*)-2(iv+ii^*)],$$

and equality holds if and only if $\mu_c(j)\leq 3$ for all c with $j(c)=0$.

2. The number $d-(iii+iii^*)$ is non-negative and multiple of 2. Moreover

$$i_0(1)+i_0^*(1)\leq\frac{1}{2}[d-(iii+iii^*)],$$

and equality holds if and only if $\mu_c(j)\leq 2$ for all c with $j(c)=1$.

Proof. The above relations give

$$(ii+iv^*)+2(iv+ii^*)+3(i_0(0)+i_0^*(0))=\sum_{m\equiv 1\ (3)}k_0(m)+2\sum_{m\equiv 2\ (3)}k_0(m)+3\sum_{m\equiv 0\ (3)}k_0(m)\leq d,$$

and more precisely

$$\begin{aligned}d-(ii+iv^*)-2(iv+ii^*)-3(i_0(0)+i_0^*(0))&=\\&=\sum_{m\equiv 1\ (3)}(m-1)k_0(m)+\sum_{m\equiv 2\ (3)}(m-2)k_0(m)+\sum_{m\equiv 0\ (3)}(m-3)k_0(m).\end{aligned}$$

Therefore $d-(ii+iv^*)-2(iv+ii^*)$ is non-negative and multiple of 3. The inequality and the equivalence in point 1. are now obvious. We can deal with the second point exactly in the same way. □

Denote with R_{j_0} the multiplicity of j_0 in Hurwitz's formula, i.e.

$$R_{j_0}=\sum_{m\geq 1}(m-1)k_{j_0}(m)=d-\sum_{m\geq 1}k_{j_0}(m).$$

Corollary 2.2.25. *1. The number $2d-2(ii+iv^*)-(iv+ii^*)$ is multiple of 3. Moreover*

$$R_0\geq\frac{1}{3}[2d-2(ii+iv^*)-(iv+ii^*)],$$

and equality holds if and only if $\mu_c(j)\leq 3$ for all c with $j(c)=0$.

2. The number $d-(iii+iii^*)$ is multiple of 2. Moreover

$$R_1\geq\frac{1}{2}[d-(iii+iii^*)],$$

and equality holds if and only if $\mu_c(j)\leq 2$ for all c with $j(c)=1$.

3. We have the equality

$$R_\infty=d-\sum_{m\geq 1}(i_m+i_m^*).$$

Proof. The last point is immediate. Using the previous lemma, we have

$$\begin{aligned}R_0=d-\sum_{m\geq 1}k_0(m)&=d-\sum_{m\equiv 1\ (3)}k_0(m)-\sum_{m\equiv 2\ (3)}k_0(m)-\sum_{m\equiv 0\ (3)}k_0(m)=\\&=d-(ii+iv^*)-(iv+ii^*)-(i_0(0)+i_0^*(0))\geq\\&\geq d-(ii+iv^*)-(iv+ii^*)-\frac{1}{3}[d-(ii+iv^*)-2(iv+ii^*)]=\\&=\frac{1}{3}[2d-2(ii+iv^*)-(iv+ii^*)],\end{aligned}$$

and the equivalence descends immediately from the equivalence of the lemma. The second point is analogous. □

Put

$$x = 2g - 2 + \frac{1}{6}\left[6\sum_{m\geq 1}(i_m + i_m^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - d\right],$$

where $g = g(C)$ is the genus of the base curve C. Then the previous computations can be combined together to obtain the next interesting result about the ramification of the j-map:

Theorem 2.2.26. *x is a non-negative integer. Moreover $x = 0$ if and only if $\mu_c(j) \leq 3$ for all c with $j(c) = 0$, $\mu_c(j) \leq 2$ for all c with $j(c) = 1$, and the map j is only ramified over $0, 1, \infty$.*

Proof. If $R' = R - R_0 - R_1 - R_\infty$ is the ramification of j away from $0, 1, \infty$, then Hurwitz's formula gives

$$\begin{aligned} 2g - 2 &= -2d + R \geq -2d + R_0 + R_1 + R_\infty \geq \\ &\geq -2d + \frac{1}{3}[2d - 2(ii + iv^*) - (iv + ii^*)] + \frac{1}{2}[d - (iii + iii^*)] + \left[d - \sum_{m\geq 1}(i_m + i_m^*)\right] = \\ &= \frac{1}{6}d - \sum_{m\geq 1}(i_m + i_m^*) - \frac{2}{3}(ii + iv^*) - \frac{1}{3}(iv + ii^*) - \frac{1}{2}(iii + iii^*), \end{aligned}$$

which is exactly the desired inequality $x \geq 0$. Moreover, $x = 0$ if and only if the two inequalities are equalities, and this happens if and only if $R' = 0$, $\mu_c(j) \leq 3$ for all c with $j(c) = 0$ and $\mu_c(j) \leq 2$ for all c with $j(c) = 1$. □

Hence we can use the number x to identify the ramification away from $0, 1, \infty$; we will call this "*extra*" ramification. In particular, using Corollary 2.2.25, we have that if there is no extra ramification, the ramification of the j-map is completely determined by the singular fibers. We will say that the elliptic surface $\pi\colon X \to C$ *has no extra j-ramification* if the j-map is not constant and $x = 0$.

2.3 Numerical Invariants

In this short section we compute all the standard invariants for a Weierstrass fibration. Actually, this could have come much earlier in the exposition, but since we hadn't an urgent need for it, we have decided to postpone it a bit. In the following, let $\pi\colon X \to C$ be a Weierstrass fibration.

Proposition 2.3.1. *The irregularity $q = h^1(X, \mathcal{O}_X)$ of the elliptic surface X is*

$$q = \begin{cases} g+1 & \text{if } X \text{ is a product} \\ g & \text{otherwise} \end{cases}$$

where $g = g(C)$ is the genus of the base curve C.

Proof. Consider the Leray spectral sequence $E_2^{pq} = H^p(C, R^q\pi_*\mathcal{O}_X)$ abutting to $H^{p+q}(X, \mathcal{O}_X)$. The low terms exact sequence is

$$0 \longrightarrow H^1(C, \pi_*\mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^0(C, R^1\pi_*\mathcal{O}_X) \longrightarrow H^2(C, \pi_*\mathcal{O}_X) = 0,$$

hence $q = h^1(C, \mathcal{O}_C) + h^0(C, \mathscr{L}^{-1}) = g + h^0(C, \mathscr{L}^{-1})$ and we conclude applying Proposition 2.2.13. □

Proposition 2.3.2. *The geometric genus $p_g = h^0(X, K_X)$ of the elliptic surface X is*

$$p_g = \begin{cases} g + \deg(\mathscr{L}) & \text{if } X \text{ is a product} \\ g + \deg(\mathscr{L}) - 1 & \text{otherwise} \end{cases}$$

Proof. By Proposition 2.2.11 we have $K_X = \pi^*(K_C \otimes \mathscr{L})$, and

$$\pi_*K_X = \pi_*(\pi^*(K_C \otimes \mathscr{L}) \otimes \mathcal{O}_X) = (K_C \otimes \mathscr{L}) \otimes \mathcal{O}_C = K_C \otimes \mathscr{L}$$

by the projection formula. Therefore

$$p_g = h^0(X, \pi^*(K_C \otimes \mathscr{L})) = h^0(C, K_C \otimes \mathscr{L}) = h^1(C, \mathscr{L}^{-1})$$

by Serre duality on C. Now we apply Riemann Roch to the last term, obtaining

$$p_g = h^0(C, \mathscr{L}^{-1}) - \deg(\mathscr{L}^{-1}) + g - 1 = h^0(C, \mathscr{L}^{-1}) + \deg(\mathscr{L}) + g - 1,$$

and we conclude again by Proposition 2.2.13. □

Recall that the Euler characteristic of the sheaf $\mathcal{O}_X$ is $\chi(\mathcal{O}_X) = 1 - q + p_g$, and Noether's formula gives:

Corollary 2.3.3. *For every elliptic surface X we have*

$$\chi(\mathcal{O}_X) = \deg(\mathscr{L}) \qquad \text{and} \qquad \chi_{top}(X) = 12\deg(\mathscr{L}).$$

Incidentally, we have also computed all the Hodge numbers, except for $h^{1,1}(X)$. But the Euler characteristic $\chi_{top}(X)$ is by definition the alternating sum of the Betti numbers of X, which in turn are sum of Hodge numbers. Explicitly, $h^{1,1}(X) = \chi_{top}(X) - 2 + 4h^{1,0}(X) - 2h^{2,0}(X)$, hence

$$h^{1,1}(X) = \begin{cases} 10\deg(\mathscr{L}) + 2g + 2 & \text{if } X \text{ is a product} \\ 10\deg(\mathscr{L}) + 2g & \text{otherwise} \end{cases}$$

At this point we want to remark that we could have derived this formula for χ_{top} using all our work on Weierstrass fibrations. First of all, Corollary 2.2.9 says that the number $12\deg\mathscr{L}$ coincides with the degree of the divisor $\{\Delta = 0\}$ on X and with the number of singular fibers (counted with multiplicities). On the other hand, we obtained a "local" version of the desired equality in Table 2.4, when we noticed that $\chi = \delta$ in all cases. Now, the degree of the divisor $\{\Delta = 0\}$ is precisely the sum of the (local) orders of vanishing δ around the singular fibers, and the same holds for χ_{top}:

Lemma 2.3.4. *Let $X_1, \ldots, X_k$ be the singular fibers. Then*

$$\chi_{top}(X) = \sum_{i=1}^{k} \chi_{top}(X_i).$$

Proof. If $S = \pi(X_1 \cup \ldots \cup X_k) \subseteq C$, then $\pi_1 \colon X \backslash \pi^{-1}(S) \to \mathbb{C} \backslash S$ is a topological covering, so the Euler characteristic $\chi_{top}(X \backslash \pi^{-1}(S))$ is the product of $\chi_{top}(C \backslash S)$ with the Euler characteristic of any fiber. Since any fiber of π_1 is a smooth elliptic curve, we get

$$\chi_{top}(X) = \chi_{top}(X \backslash \pi^{-1}(S)) + \sum_{i=1}^{k} \chi_{top}(X_i) = \sum_{i=1}^{k} \chi_{top}(X_i).$$

$\square$

Therefore, we managed to obtain the "global" equality

$$\chi_{top}(X) = 12 \deg(\mathscr{L}) = \deg(\{\Delta = 0\})$$

from its "local version".

To complete the standard list of invariants for X, it remains to compute the plurigenera of X.

Proposition 2.3.5. *The plurigenera $P_n = h^0(X, K_X^n)$ of the elliptic surface X are*

$$P_n = n(2g - 2 + \deg(\mathscr{L})) + 1 - g + h^0(C, K_C^{1-n} \otimes \mathscr{L}^{-n}).$$

Proof. Reasoning as in Proposition 2.3.2, we have

$$P_n = h^0(C, K_C^n \otimes \mathscr{L}^n) = \deg(K_C^n \otimes \mathscr{L}^n) + 1 - g + h^1(C, K_C^n \otimes \mathscr{L}^n),$$

and we reach the desired formula applying Serre duality to the last term. $\square$

Corollary 2.3.6. *Let $n \geq 2$.*

1. *If $g = 0$, then*

$$P_n = \begin{cases} 0 & \textit{if } \deg(\mathscr{L}) \leq 1 \\ 1 + n(\deg(\mathscr{L}) - 2) & \textit{otherwise} \end{cases}$$

2. *If $g = 1$, then*

$$P_n = \begin{cases} 0 & \textit{if } \deg(\mathscr{L}) = 0 \textit{ and } \operatorname{ord}(\mathscr{L}) \nmid n \\ 1 & \textit{if } \deg(\mathscr{L}) = 0 \textit{ and } \operatorname{ord}(\mathscr{L}) \mid n \\ n \deg(\mathscr{L}) & \textit{if } \deg(\mathscr{L}) \geq 1 \end{cases}$$

3. *If $g \geq 2$, then*

$$P_n = n(2g - 2 + \deg(\mathscr{L})) + 1 - g.$$

Proof. It is all quite clear: if $g = 0$ and $\deg(\mathscr{L}) \leq 1$, then $K_C \otimes \mathscr{L}$ has negative degree, and so $P_n = h^0(C, K_C^n \otimes \mathscr{L}^n) = 0$; instead, if $\deg(\mathscr{L}) \geq 1$, we apply the previous proposition, noticing that $\deg(K_C^{1-n} \otimes \mathscr{L}^{-n}) \leq -2$. The case $g = 1$ is immediate: recall that, if $\deg(\mathscr{L}) = 0$, then it is a torsion line bundle, and a line bundle of degree 0 has a section if and only if it is 0. Finally, the third point is easy using that $\deg(K_C) > 0$ and $\deg(\mathscr{L}) \geq 0$. $\square$

Now we are ready to classify elliptic surfaces according to the genus of the base curve. We will say that X is a *properly elliptic surface* if its Kodaira dimension is 1. Moreover, an elliptic surface is called *bielliptic* if it is of the form $X = (E_1 \times E_2)/G$, where E_1, E_2 are elliptic curves and G is a finite group of translations of E_1 acting on E_2 such that $E_2/G = \mathbb{P}^1$. There exists a well known list of all possible bielliptic surfaces, due to Bagnera-de Franchis (see for example [Bea96, List VI.20]); they share the property that the canonical bundle is a torsion bundle of order $2, 3, 4$ or 6, and in particular $12K_X$ is trivial.

Corollary 2.3.7. *1. If $g = 0$, then*

$$X \text{ is a } \begin{cases} \textit{product } X_\eta \times \mathbb{P}^1 & \textit{if } \deg(\mathscr{L}) = 0 \\ \textit{rational surface} & \textit{if } \deg(\mathscr{L}) = 1 \\ \textit{K3 surface} & \textit{if } \deg(\mathscr{L}) = 2 \\ \textit{properly elliptic surface} & \textit{otherwise} \end{cases}$$

2. If $g = 1$, then

$$X \text{ is a } \begin{cases} \textit{product (i.e. an abelian surface)} & \textit{if } \mathscr{L} = \mathcal{O}_C \\ \textit{bielliptic surface} & \textit{if } \deg(\mathscr{L}) = 0 \textit{ and } \operatorname{ord}(\mathscr{L}) > 1 \\ \textit{properly elliptic surface} & \textit{otherwise} \end{cases}$$

It follows from [Bea96, List VI.20] that, if X is bielliptic, then $\operatorname{ord}(K_X) = \operatorname{ord}(\mathscr{L})$ *(recall that in this case $12\mathscr{L}$ is trivial).*

3. If $g \geq 2$, then X is a properly elliptic surface.

As a consequence, we have that if X is rational, necessarily $g = q = 0$, and so C must be rational, too. Moreover, the last corollary says that $\mathscr{L}$ is forced to have degree 1, hence $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1}(1)$. As we have already noticed, all the examples we have given of rational elliptic surfaces come from pencils of plane curves, and in fact they have $\deg(\mathscr{L}) = 1$. Actually, these are the only possible rational elliptic surfaces, as the next result shows. Recall the definition of the Hirzebruch surfaces

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) :$$

they are the minimal rational models (except for $n = 1$) and each of them has a unique exceptional curve, of self-intersection $-n$.

Theorem 2.3.8. *Let $\pi : X \to C$ be a rational elliptic surface with section. Then $C = \mathbb{P}^1$, π is induced by a pencil of generically smooth plane cubics, and X is the blow-up of the 9 base points of the pencil.*

Proof. Consider a minimal model Y of X, and denote by $\eta : X \to Y$ the appropriate blow down to get to Y. From Example 2.1.5 we know that $K_X = -F$, where F is any fiber of π; thus any exceptional smooth rational curve E on X has $E^2 \geq -2$, since by the genus formula

$$0 = g(E) = 1 + \frac{E^2 + EK_X}{2} = \frac{E^2 - EF}{2},$$

and $EF \geq 0$ varying appropriately F. Therefore Y can be only be $\mathbb{P}^2$, $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$, because the self-intersection strictly decreases upon blow-ups; we want to see that we can choose $Y = \mathbb{P}^2$.
Suppose $Y = \mathbb{F}_0$; then any blow-up of Y also dominates $\mathbb{P}^2$ (remember that if we blow up $\mathbb{P}^2$ in two points and blow down the line joining them we obtain $\mathbb{F}_0$), and we're done.
Suppose instead $Y = \mathbb{F}_2$; if η contains the blow-up of a point in the exceptional curve of Y, then its strict transform in X has self-intersection ≤ -3, a contradiction. Hence η blows up a point p away from the (horizontal) exceptional curve C. Let F_p be the fiber of $\mathbb{F}_2$ containing p, $\widetilde{F}$ its strict

transform and E the exceptional divisor. Clearly $\widetilde{F}^2 = E^2 = -1$, so we can blow down $\widetilde{F}$; after this, the self intersection of C becomes -1. Consequently, the blow-up of $\mathbb{F}_2$ dominates $\mathbb{P}^2$, since the lowest self-intersection is -1.
Therefore $Y = \mathbb{P}^2$. The pencil $|F| = |-K_X|$ on X giving the map π descends to a pencil of generically smooth curves contained in $|-K_{\mathbb{P}^2}| = |\mathcal{O}_{\mathbb{P}^2}(3)|$, i.e. a pencil of generically smooth plane curves. $\square$

2.4 Quadratic Twists

In this section we continue the study of the local aspects of Weierstrass fibrations around the singular fibers, and we introduce an incredibly useful machinery that allows us to deform a given elliptic surface preserving its j-map. Recall that we have denoted by $\mathbb{BW}$ the set of Weierstrass data, up to birational equivalence.

We begin with a simple remark: if two Weierstrass data $(\mathscr{L}_1, A_1, B_1)$, $(\mathscr{L}_2, A_2, B_2)$ induce birational Weierstrass fibrations over C, then the two induced j-maps $j_1, j_2 \colon C \to \mathbb{P}^1$ concide. Indeed, by Proposition 2.2.18, there exist line bundles M_1, M_2 on C and sections $f_1 \in H^0(M_1)$, $f_2 \in H^0(M_2)$ such that the Weierstrass data $(\mathscr{L}_1 \otimes M_1, A_1 f_1^4, B_1 f_1^6)$, $(\mathscr{L}_2 \otimes M_2, A_2 f_2^4, B_2 f_2^6)$ are isomorphic. Now just notice that

$$j(A_i f_i^4, B_i f_i^6) = \frac{4A_i^3 f_i^{12}}{4A_i^3 f_i^{12} + 27 B_i^2 f_i^{12}} = j(A_i, B_i),$$

and we have proved the remark. We can actually do better than this:

Proposition 2.4.1. *Two Weierstrass data $(\mathscr{L}_1, A_1, B_1)$, $(\mathscr{L}_2, A_2, B_2)$ induce the same j-map (not identically 0 or 1) if and only if there exist line bundles M_1, M_2 on C and non-zero sections $f_1 \in H^0(M_1^2)$, $f_2 \in H^0(M_2^2)$ such that the Weierstrass data $(\mathscr{L}_1 \otimes M_1, A_1 f_1^2, B_1 f_1^3)$, $(\mathscr{L}_2 \otimes M_2, A_2 f_2^2, B_2 f_2^3)$ are isomorphic.*

Proof. One implication is immediate, since $j(A_i f_i^2, B_i f_i^3) = j(A_i, B_i)$ as above. Conversely, since $j_1 = j_2$ are not identically 0 or 1, the A_i and B_i are not identically 0, hence $A_1^3 B_2^2 = A_2^3 B_1^2$ (this equality holds locally, and we can extend it as an equality between sections of $\mathscr{L}^{24}$). We consider the line bundles $M_1 = \mathscr{L}_1^2 \otimes \mathscr{L}_2^3$, $M_2 = \mathscr{L}_1^3 \otimes \mathscr{L}_2^2$ and the non-zero sections $f_1 = A_1 B_2$, $f_2 = A_2 B_1$ respectively of M_1^2 and M_2^2: it is immediate to verify that they are the desired line bundles and sections. □

Concretely, this criterion is the counterpart for elliptic surfaces of Lemma 1.1.5, where A, B are sections rather than numbers.

Definition 2.4.2. A *quadratic twist* on the Weierstrass data $(\mathscr{L}, A, B)$ is the substitution of this data with the new data $(\mathscr{L} \otimes M, Af^2, Bf^3)$ for some line bundle M on C and some non-zero section $f \in H^0(M^2)$.

If we apply twice a quadratic twist to a set of Weierstrass data, we obtain a birational set of data. Therefore the quadratic twists are involutions in the set $\mathbb{BW}$, and they identify Weierstrass data with the same j-map, assumed that this map is not identically 0 or 1.

Notice for instance that bielliptic surfaces with $\operatorname{ord}(K_C) = 2$ are equivalent to product elliptic surfaces up to quadratic twists: if $(\mathscr{L}, A, B)$, with $\mathscr{L}^2 = \mathcal{O}_C$, is a set of Weierstrass data inducing a bielliptic surface (with $j(A, B) \not\equiv 0, 1$), we can perform a quadratic twist with $M = \mathscr{L}$ and $f = 1$, and obtain the Weierstrass data $(\mathcal{O}_C, A, B)$, inducing a product surface.

In order to work explicitly with quadratic twists, we have to understand better their action on the set $\mathbb{BW}$. We will do this a more general setting, and then we will use the main ideas in our specific case.

Let C be a curve, not necessarily compact, and S any subset of C.

Definition 2.4.3. A *double cover pair* on C relative to S is a pair (M, f), where M is a line bundle over C and f a non-zero section of M^2, with zeroes contained in S.

We say that two double cover pairs (M_1, f_1), (M_2, f_2) are *isomorphic* if there exists an isomorphism $\phi \colon M_1 \to M_2$ such that ϕ^2 carries the section $f_1 \in H^0(M_1^2)$ into $f_2 \in H^0(M_2^2)$. The quotient of all double cover pairs relative to S up to isomorphism is denoted by $\mathscr{A}_S$; it is endowed with the obvious product

$$[M_1, f_1] \cdot [M_2, f_2] = [M_1 \otimes M_2, f_1 f_2],$$

where we denote by $[M, f]$ the equivalence class $[(M, f)]$ of (M, f). Since the usual and tensor products are commutative and associative, so it is this product on $\mathscr{A}_S$; moreover $[\mathcal{O}_C, 1]$ is its neutral element.

The analogy with our case is quite clear; our goal is to discard those double cover pairs acting trivially on $\mathbb{BW}$, so it is natural to quotient for the subset $\mathscr{B}_S \subseteq \mathscr{A}_S$ formed by the classes $[M, f^2]$, where f is a non-zero section of M with zeroes contained in S. Notice that it is legal to quotient for $\mathscr{B}_S$, since it contains the neutral element $[\mathcal{O}_C, 1]$, and the product $[M_1, f_1^2] \cdot [M_2, f_2^2] = [M_1 \otimes M_2, f_1^2 f_2^2]$ is again an element of $\mathscr{B}_S$. Concretely, we identify two elements $[M_1, f_1], [M_2, f_2] \in \mathscr{A}_S$ if and only if there exist two elements $[N_1, g_1^2], [N_2, g_2^2] \in \mathscr{B}_S$ such that $[M_1, f_1] \cdot [N_1, g_1^2] = [M_2, f_2] \cdot [N_2, g_2^2]$.

Definition 2.4.4. The *double cover group* relative to S is the quotient $\mathrm{Doub}_S(C) = \mathscr{A}_S/\mathscr{B}_S$ with the induced product. If $S = C$, we could drop the subscript S and simply write $\mathrm{Doub}(C)$.

It is actually a bit early to call the set $\mathrm{Doub}_S(C)$ *group*; however, it is immediate to see that $[M, f]^2 = [M^2, f^2] \in \mathscr{B}_S$ for every $[M, f] \in \mathscr{A}_S$, hence $\mathrm{Doub}_S(C)$ it is not simply a group, but even a *boolean group*, i.e. a group whose non-trivial elements have order 2. The identity of $\mathrm{Doub}_S(C)$ is the class $[\mathcal{O}_C, 1]$.

This construction was motivated by our need to study the action of quadratic twists on $\mathbb{BW}$; however, we can interpret the group $\mathrm{Doub}_S(C)$ in an alternative way, equally important in the following. Let C_1 be a reduced curve, and choose a double covering $\varphi\colon C_1 \to C$. Then, by Lemma 1.3.21, the direct image $\varphi_* \mathcal{O}_{C_1}$ is isomorphic to the sheaf $\mathcal{O}_C \oplus M^{-1}$, where M is the sheaf over C inducing the double covering φ. Locally, there exists a section s of M such that $s^2 = f$ for some section f of $\mathcal{O}_C$, i.e. a local splitting of the direct image $\varphi_* \mathcal{O}_{C_1}$. Therefore f can be seen as a section of M^2 by patching together the local s^2; if f were zero, there would have been nilpotents on C_1, a contradiction since C_1 is reduced. Moreover, φ is branched precisely over the (local) set $\{f = 0\}$.

We have just described a map from the set of (reduced) double coverings of C branched over a set $B \subseteq S$ to the set of double cover pairs on C relative to S; repeating the argument backwards, we realize that this correspondence is actually bijective. It is interesting to understand which effect have the two consecutive quotients (to get $\mathscr{A}_S$ and $\mathrm{Doub}_S(C)$), seeing the double cover pairs as double coverings.

It is clear that two double coverings are isomorphic if and only if the corresponding double cover pairs are isomorphic; on the other hand, when we quotient for $\mathscr{B}_S$, we are identifying birational double coverings: a double covering φ given by $[M, f^2] \in \mathscr{B}_S$ globally splits, in the sense that it is globally defined by the equation $t^2 = f^2$, and so its normalization must be the trivial covering.
We can also describe the product in $\mathrm{Doub}_S(C)$ in terms of double coverings: if $\varphi_1\colon C_1 \to C$, $\varphi_2\colon C_2 \to C$ are two double coverings with involutions ι_1, ι_2 (that locally send t to $-t$), then we can consider the cartesian diagram

$$\begin{tikzcd} C_1 \times_C C_2 \arrow[r, "\widetilde{\varphi}_2"] \arrow[d, "\widetilde{\varphi}_1"'] \arrow[dr, "\varphi"] & C_1 \arrow[d, "\varphi_1"] \\ C_2 \arrow[r, "\varphi_2"'] & C \end{tikzcd}$$

and use the universal property of the pull-back to lift ι_1, ι_2 to an involution $\widetilde{\iota}$ of $C_1 \times_C C_2$ obtained by

$$\begin{tikzcd}
C_1 \times_C C_2 \arrow[dr, dashed, "\widetilde{\iota}"] \arrow[drr, bend left, "\iota_1 \circ \widetilde{\varphi}_2"] \arrow[ddr, bend right, "\iota_2 \circ \widetilde{\varphi}_1"'] & & \\
& C_1 \times_C C_2 \arrow[r, "\widetilde{\varphi}_2"] \arrow[d, "\widetilde{\varphi}_1"'] & C_1 \arrow[d, "\varphi_1"] \\
& C_2 \arrow[r, "\varphi_2"'] & C
\end{tikzcd}$$

Then it is easy to see that the product of the double coverings φ_1, φ_2 is exactly the double covering φ; clearly this is a product in $\mathscr{A}_S$, and it descends to $\mathrm{Doub}_S(C)$ (seen as the set of the double coverings over C) after the identification of birational coverings. Thus we have a 1-1 correspondence

$$\mathrm{Doub}_S(C) \longleftrightarrow \left\{ \begin{matrix} \text{reduced double coverings over } C \\ \text{with branch locus} \subseteq S \end{matrix} \right\}_{/\sim}$$

(where $\sim$ denotes the birational equivalence) that is actually a group isomorphism.

Consequently, we would like to transfer the properties we know about coverings of curves to the group $\mathrm{Doub}_S(C)$. For instance, up to birational equivalence we can only consider smooth double coverings, since each double covering admits a unique smooth model (concretely, it is the normalization of the covering, because a normal curve is smooth); and smooth double coverings have a reduced branch locus, so we would like to pinpoint, for each element $[M, f] \in \mathrm{Doub}_S(C)$, a "smooth model" corresponding to the smooth model of the double covering induced by $[M, f]$. Amazingly, this is actually possible:

Proposition 2.4.5. *Each element $[M, f] \in \mathrm{Doub}_S(C)$ has a unique representative (which we will also denote by $[M, f]$) in $\mathscr{A}_S$ with reduced divisor of zeroes $(f)_0$.*

Proof. The existence is not difficult: if $[M, f]$ is any element in $\mathscr{A}_S$, write $(f)_0 = D + 2E$, with $D, E \geq 0$ and D reduced. We want to throw away the non-reduced part $2E$. Therefore consider the line bundle $\mathcal{O}_C(E)$ and a section s of $\mathcal{O}_C(E)$ vanishing exactly on E: the birational double cover pair given by $\left[M(-E), \frac{f}{s^2}\right]$ has a reduced divisor of zeroes (equal to D), and therefore it is a "smooth" representative for $[M, f] \in \mathrm{Doub}_S(C)$.
Now we prove the uniqueness: assume that $[M_1, f_1]$ and $[M_2, f_2]$ are birational double cover pairs such that $D_1 = (f_1)_0$ and $D_2 = (f_2)_0$ are reduced. By definition, there exist line bundles N_1, N_2 on C and sections g_1, g_2 respectively of N_1^2 and N_2^2 such that $[M_1 \otimes N_1, f_1 g_1^2] = [M_2 \otimes N_2, f_2 g_2^2]$. If $E_i = (g_i)_0$ for $i = 1, 2$, then the last equality forces $D_1 + 2E_1 = D_2 + 2E_2$, where $D_1, D_2, E_1, E_2 \geq 0$. But since the D_i are reduced, this decompositions must coincide, i.e. $D_1 = D_2$ and $E_1 = E_2$. Therefore $N_1 \cong N_2$, and we can choose the isomorphism between them to send g_1 into g_2, seeing that g_1 and g_2 have the exact same zeroes. Therefore we can tensor the equality $[M_1 \otimes N_1, f_1 g_1^2] = [M_2 \otimes N_2, f_2 g_2^2]$ by the equality $\left[N_1^{-1}, \frac{1}{g_1^2}\right] = \left[N_2^{-1}, \frac{1}{g_2^2}\right]$ to get $[M_1, f_1] = [M_2, f_2]$. $\square$

Consequently, we have refined the above correspondence, obtaining a 1-1 correspondence

$$\mathrm{Doub}_S(C) = \{[M, f] \in \mathscr{A}_S \mid (f)_0 \text{ is reduced}\} \longleftrightarrow \left\{ \begin{matrix} \text{smooth double coverings over } C \\ \text{with branched locus} \subseteq S \end{matrix} \right\}$$

Now we can return to our Weierstrass fibration $\pi \colon X \to C$, where C is a projective curve. We will often assume S finite, because S will often be the set of points $c \in C$ such that the fiber X_c is singular; in this special case we have a precise description of the group $\mathrm{Doub}_S(C)$. First, a general lemma:

Lemma 2.4.6. *Let C be a projective curve of genus g, and L a line bundle on C with even degree. Then there exist 2^{2g} line bundles M on C such that $L = M^2$. These bundles M are called the square roots of the line bundle L.*

Proof. Consider the exponential sequence

$$0 \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}(C) \xrightarrow{c_1} H^2(C,\mathbb{Z}) = \mathbb{Z} \longrightarrow 0;$$

recall that $\mathrm{Pic}^0(C) = \mathrm{Jac}(C)$ is a complex torus of complex dimension g, and the Chern map is simply the degree. The surjectivity of c_1 gives a line bundle M' on C such that $2\deg(M') = \deg(L)$, i.e. $L \otimes M'^{-2}$ has degree 0. By the exactness of the exponential sequence, $L \otimes M'^{-2} \in \mathrm{Pic}^0(X)$. The number of square roots of L coincides with the number of square roots of $L \otimes M'^{-2}$, which is 2^{2g}: choose any element M'' in $\mathrm{Pic}^0(X)$ such that $M''^2 = L \otimes M'^{-2}$, and the other square roots come from tensoring M'' by the 2-torsion in $\mathrm{Pic}^0(C)$. ☐

Proposition 2.4.7. *Let C be a projective curve of genus g, and $S \subseteq C$ finite. Then the group* $\mathrm{Doub}_S(C)$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^n$, *where*

$$n = \begin{cases} 2g & \text{if } S = \emptyset \\ 2g + |S| - 1 & \text{otherwise} \end{cases}$$

Proof. To compute the cardinality of $\mathrm{Doub}_S(C)$, we have to count how many double cover pairs $[M,f]$ exist with reduced divisor of zeroes $(f)_0$ contained inside S. First of all, the degree of the divisor $(f)_0$ is even, because f is a section of M^2. Therefore, if S is empty, there is one possible divisor $(f)_0$ only, i.e. the divisor 0. If instead S is nonempty, we have $\binom{|S|}{2i}$ possibilities for a divisor $(f)_0$ of degree $2i$, and each of them uniquely determines a line bundle L on C of even degree. By the previous lemma, we have 2^{2g} choices for the square root M of L, end each choice gives a double cover pair $[M,f]$. Summing up, we have 2^{2g} possibilities if $S = \emptyset$ and

$$2^{2g} \sum_{i \geq 0} \binom{|S|}{2i} = 2^{2g} 2^{|S|-1}$$

possibilities if S is nonempty. We conclude using that $\mathrm{Doub}_S(C)$ is boolean, and the classification of finite abelian groups. ☐

Now we let the group $\mathrm{Doub}_S(C)$ act on $\mathbb{BW}$: if $(\mathscr{L}, A, B)$ is a set of Weierstrass data for a Weierstrass fibration, and $[M,f] \in \mathrm{Doub}_S(C)$ is a double cover pair, then we define

$$[M,f] \cdot (\mathscr{L}, A, B) = (\mathscr{L} \otimes M, Af^2, Bf^3).$$

Clearly this action descends to an action of $\mathrm{Doub}_S(C)$ on $\mathbb{BW}$: just apply Proposition 2.2.18.
Thus this action is equivalent to performing quadratic twists on Weierstrass fibration, and the fact that every non-trivial element of $\mathrm{Doub}_S(C)$ has order 2 reflects the fact that the double execution of a quadratic twist produces birational Weierstrass fibrations.

Proposition 2.4.8. *The action of* $\mathrm{Doub}_S(C)$ *on* $\mathbb{BW}$ *is free, i.e. every non-trivial* $[M,f] \in \mathrm{Doub}_S(C)$ *fixes no element in* $\mathbb{BW}$.

Proof. Suppose that $[M,f] \cdot (\mathscr{L}, A, B) = (\mathscr{L}, A, B)$, i.e. $(\mathscr{L} \otimes M, Af^2, Bf^3) = (\mathscr{L}, A, B)$ as elements of $\mathbb{BW}$. Therefore there exist line bundles N_1, N_2 on C and non-zero sections g_1, g_2 of N_1, N_2 such that $(\mathscr{L} \otimes M \otimes N_1, Af^2 g_1^4, Bf^3 g_1^6) = (\mathscr{L} \otimes N_2, Ag_2^4, Bg_2^6)$ as Weierstrass data. Therefore there exists a bundle isomorphism $\phi\colon \mathscr{L} \otimes M \otimes N_1 \to \mathscr{L} \otimes N_2$ such that ϕ^4 sends $Af^2 g_1^4$ into Ag_2^4 and ϕ^6 sends $Bf^3 g_1^6$ into Bg_2^6. Tensoring ϕ by the identity $\mathscr{L}^{-1} \otimes N_1^{-1} \to \mathscr{L}^{-1} \otimes N_1^{-1}$, we obtain a second isomorphism $\psi\colon M \to N_2 \otimes N_1^{-1}$ such that ψ^4 sends f^2 into $\frac{g_2^4}{g_1^4}$ and ψ^6 sends f^3 into $\frac{g_2^6}{g_1^6}$. Interpreting $\frac{g_2}{g_1}$ as a (non-zero) section of $N_2 \otimes N_1^{-1}$, we have that necessarily ψ^2 sends f into $\frac{g_2^2}{g_1^2}$, i.e. there exists an isomorphism of Weierstrass data

$$[M,f] = \left[N_2 \otimes N_1, \frac{g_2^2}{g_1^2} \right] \in \mathscr{B}_S.$$

☐

Denote with $\mathbb{BW}^*$ the set of equivalence classes of Weierstrass data over C having a j-map not identically 0 or 1. The theory just developed can be rephrased as follows: there exists a map

$$\mathrm{j}\colon \mathbb{BW}^* \longrightarrow \{j\colon C \to \mathbb{P}^1 \text{ not identically 0 or 1}\},$$

and two elements in $\mathbb{BW}^*$ have the same image under j if and only if they are in the same orbit of $\mathbb{BW}^*$ under the action of $\mathrm{Doub}(C)$. The following easy computation gives that this map j is even surjective:

Proposition 2.4.9. *Let $j = [t, s]\colon C \to \mathbb{P}^1$ be any non-costant map, where we regard t, s as coordinates in $\mathbb{P}^1$, i.e. as sections of $\mathscr{L} = j^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then the Weierstrass data given by*

$$(\mathscr{L}, -3t(t-s)s^2, 2t(t-s)^2s^3)$$

has precisely j as j-map.

Proof. A direct computation gives

$$\begin{aligned}[4A^3, \Delta] &= \left[4(-3t(t-s)s^2)^3, 4(-3t(t-s)s^2)^3 + 27(2t(t-s)^2s^3)^2\right] = \\ &= \left[-108t^3(t-s)^3s^6, -108t^3(t-s)^3s^6 + 108t^2(t-s)^4s^6\right] = \\ &= [-t, -t + (t-s)] = [t, s].\end{aligned}$$

□

Remark 2.4.10. We can extend the map j to

$$\mathrm{j}\colon \mathbb{BW} \longrightarrow \{j\colon C \to \mathbb{P}^1\},$$

and this is again surjective: the product surfaces realize the costant j-maps.

Clearly the elliptic surface in Proposition 2.4.9 is the pull-back of a rational elliptic surface X having the identity $\mathbb{P}^1 \to \mathbb{P}^1$ as j-map (it is rational since $C = \mathbb{P}^1$ and $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1}(1)$). It is immediate to understand the configuration of the singular fibers of X: the above computation gives

$$\Delta = 108t^2(t-s)^3s^7,$$

hence we have 3 singular fibers, over $[0, 1]$, $[1, 1]$ and $[1, 0]$. By looking at the a, b, δ table 2.4, we realize that these 3 singular fibers are respectively of types II, III and I_1^*.

This surface will be the starting point to construct elliptic surfaces with specific properties. For instance, if $\pi\colon X \to C$ is a smooth minimal elliptic surface with Weierstrass data $(\mathscr{L}, A, B)$ and j-map $[t, s]\colon C \to \mathbb{P}^1$ not identically 0 or 1, then X and the Weierstrass fibration given by $(j^*\mathcal{O}_{\mathbb{P}^1}(1), -3t(t-s)s^2, 2t(t-s)^2s^3)$ have the same j-map. Therefore there exists an element $[M, f] \in \mathrm{Doub}(C)$, identifying a quadratic twist with $(f)_0$ reduced, such that $(\mathscr{L}, A, B)$ is birational to $(j^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes M, -3t(t-s)s^2f^2, 2t(t-s)^2s^3f^3)$. But our initial Weierstrass fibration $\pi\colon X \to C$ is in normal form, and thus there exist a divisor $D \geq 0$ on C and a section g of $\mathcal{O}_C(D)$ such that

$$(\mathscr{L}(D), Ag^4, Bg^6) = (j^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes M, -3t(t-s)s^2f^2, 2t(t-s)^2s^3f^3)$$

are equal as Weierstrass data. This remark gives a concrete algorithm to reconstruct the fibration π from its j-map only.

Up to now we have focused on the global effects of quadratic twists, i.e. how a Weierstrass fibration globally changes under the action of the group $\mathrm{Doub}_S(C)$. In the remaining part of the section we would like to investigate the local effects of this action: how does the group $\mathrm{Doub}_{\{c\}}(C)$ act on a neighbourhood of the singular fiber over $c \in C$?

Let Δ be a small (analytic) neighbourhood of 0, with local coordinate t, and restrict the Weierstrass fibration to $\pi\colon X \to \Delta$. Since the Picard group $\mathrm{Pic}(\Delta)$ is trivial, the group $\mathscr{A}_{\{0\}}$ can be identified with the set of double cover pairs of the form $[\mathcal{O}_\Delta, t^n]$, where n is any non-negative number. The only such element that is non-trivial and has a reduced divisor of zeroes is $[\mathcal{O}_\Delta, t]$, thus $\mathrm{Doub}_{\{0\}}(C) = \mathbb{Z}/2\mathbb{Z}$, with $\bar{1}$ corresponding to $[\mathcal{O}_\Delta, t]$. Let π be induced by the Weierstrass data $(\mathcal{O}_\Delta, A(t), B(t))$, where A, B are local holomorphic functions. We have often encountered the multiplication

$$[\mathcal{O}_\Delta, t] \cdot (\mathcal{O}_\Delta, A(t), B(t)) = (\mathcal{O}_\Delta, t^2 A(t), t^3 B(t))$$

in disguise: the functions A, B change in this way (actually backwards) when we take the strict transform of a (local) Weierstrass fibration whose trisection has a triple point at $(0, 0)$.

The a, b, δ table 2.4 let us substitute the local Weierstrass data $(\mathcal{O}_\Delta, A(t), B(t))$ with the triple (a, b, δ). Therefore we can write the action of $\mathrm{Doub}_{\{0\}}(\Delta)$ as

$$[\mathcal{O}_\Delta, t] \cdot (a, b, \delta) = (a + 2, b + 3, \delta + 6).$$

If we want to consider only Weierstrass fibrations in normal form, we impose $a \leq 3$ or $b \leq 5$, and we subtract $(4, 6, 12)$ from the last triple if $a \geq 2$ and $b \geq 3$. Notice that $a \geq 2$ and $b \geq 3$ if and only if the singular fiber is a "$*$-*fiber*", i.e. it is of one of the types I_n^*, IV^*, III^*, II^*. Consequently, the action of the non-trivial element of $\mathrm{Doub}_{\{0\}}(\Delta)$, i.e. the execution of the only non-trivial quadratic twist, interchanges $*$-fibers with non-$*$-fibers. We will refer to this switch as the *transfer of $*$ process*. In particular, again looking at the a, b, δ table 2.4, we have:

Original fiber	**Fiber after the transfer of $*$ process**
I_n	I_n^*
II	IV^*
III	III^*
IV	II^*

Table 2.6: Effect of the transfer of $*$ process. Since the process is an involution, the table can be read from left to right and vice versa.

Now that we have a clear understanding of what happens locally, we can try to work with all singular fibers at once. So consider again a Weierstrass fibration $\pi\colon X \to C$ in normal form, with Weierstrass data $(\mathscr{L}, A, B)$, and let $S \subseteq C$ be any finite subset of C. Put $S = \{c_1, \ldots, c_k\}$. We attach a number $n_i \in \{0, 1\}$ to each point c_i, such that the sum $\sum_{i=1}^k n_i$ is even, and consider the non-negative reduced divisor $D = \sum_{i=1}^k n_i c_i$. If f is a function with $(f)_0 = D$, we have a double cover pair $[M, f]$ for some line bundle M on C, and we let $[M, f]$ act over $(\mathscr{L}, A, B)$. After putting the resulting Weierstrass data $(\mathscr{L} \otimes M, Af^2, Bf^3)$ in normal form, we obtain a new set of Weierstrass data $(\mathscr{L}', A', B')$, giving a new Weierstrass fibration $\pi'\colon X' \to C$. By the local description above, the fibers of π and π' over c_i are equal if $n_i = 0$, and they change as in Table 2.6 if $n_i = 1$.

Consequently, we can perform an appropriate quadratic twist (with S equal to the set of points $c \in C$ such that the fiber X_c is a $*$-fiber) to obtain a minimal number of $*$-fibers on π': notice that, since the degree of D is even, the parity of $*$-fibers doesn't change between π and π', but we can obtain a new Weierstrass fibration $\pi'\colon X' \to C$ with at most 1 $*$-fiber. We will say that the original fibration $\pi\colon X \to C$ is $*$-*even* (respectively $*$-*odd*) if the number of $*$-fibers is even (respectively odd), i.e. whether π' has 0 or 1 $*$-fibers. Moreover, a Weierstrass fibration is $*$-*minimal* if it has at most 1 $*$-fiber.

Now notice that the transfer of $*$ process increases the numbers (a, b, δ) if and only if the corresponding fiber is a non-$*$-fiber; thus we would like to relate the $*$-minimality to a minimality of the sum of the local (a, b, δ)'s. This is quite straighforward: we say that a set of Weierstrass data $(\mathscr{L}, A, B)$ is *j-minimal* if the degree $\deg(\mathscr{L})$ is minimal among all Weierstrass data with the same j-map.

Proposition 2.4.11. *A set of minimal Weierstrass data $(\mathscr{L}, A, B) \in \mathbb{BW}^*$, i.e. with j-map not identically 0 or 1, is $*$-minimal if and only if it is j-minimal.*

Proof. Clearly, if $(\mathscr{L}, A, B)$ is not $*$-minimal, it contains two $*$-fibers over $c_1, c_2 \in C$, and performing a quadratic twist with $[M, f]$, where $M^2 = \mathcal{O}_C(c_1 + c_2)$ and $(f)_0 = c_1 + c_2$, we obtain the Weierstrass data (after putting it in normal form) $\left(\mathscr{L} \otimes M^{-1}, \frac{A}{f^2}, \frac{B}{f^3}\right)$, and $\deg(\mathscr{L} \otimes M^{-1}) = \deg(\mathscr{L}) - 1$.
Conversely, if $(\mathscr{L}, A, B)$ is $*$-minimal, we have two cases, corresponding to whether the induced Weierstrass fibration has 0 or 1 $*$-fibers. If every fiber is a non-$*$-fiber, the resulting Weierstrass data after the action of $[M, f] \in \mathrm{Doub}(C)$ is simply $(\mathscr{L} \otimes M, Af^2, Bf^3)$, being already in normal form, and $\deg(\mathscr{L} \otimes M) \geq \deg(\mathscr{L})$. If instead there is a unique $*$-fiber over $c \in C$, the worst case is when we perform a quadratic twist with $[M, f]$, where c is contained in the divisor of zeroes $(f)_0$. However, the needed normalization of $(\mathscr{L} \otimes M, Af^2, Bf^3)$ is given by tensoring by $\left(\mathcal{O}_C(-c), \frac{1}{g^4}, \frac{1}{g^6}\right)$, where g is a section of $\mathcal{O}_C(c)$ vanishing precisely on c, and we conclude by noticing that $\deg(\mathscr{L} \otimes M(-c)) = \deg(\mathscr{L}) + \deg(M(-c)) \geq \deg(\mathscr{L})$. $\square$

2.5 Monodromy

In this section we introduce a fundamental tool in the study of fibrations: the monodromy around singular fibers. In the special case of elliptic fibrations, the monodromy can be explicitly computed in every case, and it will help us distinguish the types of singular fibers.

It was often useful to focus on arbitrarily small neighbourhoods of singular fibers, in order to obtain a local study of their geometry. We are going to write this concept down in a more formal way, once and for all. Concretely, we would like to single out the germ of the Weierstrass fibration $\pi\colon X \to C$ around a singular fiber X_c over $c \in C$.

Definition 2.5.1. The *germ of the fiber* of X_c is the equivalence class $(\Delta, \pi|_{\pi^{-1}(\Delta)})/\sim$, where Δ is any (analytic) neighbourhood of c and $(\Delta_1, \pi 1|_{\pi_1^{-1}(\Delta_1)}) \sim (\Delta_2, \pi 2|_{\pi_2^{-1}(\Delta_2)})$ if there exists another neighbourhood $\Delta_3 \subseteq \Delta_1 \cap \Delta_2$ of c such that the fibrations $\pi 1|_{\pi_1^{-1}(\Delta_3)}$ and $\pi 2|_{\pi_2^{-1}(\Delta_3)}$ are isomorphic.

In many cases we were only interested in the germ of a fiber, rather than in the whole fibration. The advantage of working locally is that the germ of a fiber is identified by very little data.

Proposition 2.5.2. *The germ of a fiber X_c is uniquely determined by $j(c)$, $\mu_c(j)$ and the type of singular fiber of X_c.*

Proof. Clearly, the germ of the map $j\colon C \to \mathbb{P}^1$ around c is uniquely determined by $j(c)$ and $\mu_c(j)$. If we choose a sufficiently small neighbourhood Δ of c, we can assume that X_c is the only singular fiber over Δ. If j is not identically 0 or 1, we remain with two possibilities for the germ of the fiber, parametrized by the group $\mathrm{Doub}_{\{c\}}(\Delta)$. Since the action of this group interchanges $*$-fibers with non-$*$-fibers, the knowledge of the type of singular fiber over c let us recognize which one occurs.
Now, if j is identically 0 on Δ, the fibration can be (locally) written $y^2 = x^3 + t^b$ for some $0 \le b \le 5$. Looking at the a, b, δ table 2.4, we see that the 6 possibilities for b correspond to 6 different types of singular fiber (respectively I_0, II, IV, I_0^*, IV^*, II^*), and the knowledge of this type for X_c concludes the identification. Similarly, if j is identically 1 on Δ, the fibration locally looks as $y^2 = x^3 + t^a x$ for some $0 \le a \le 3$, and each a uniquely determines the type of singular fiber. □

In the following, let Δ be an analytic disc, with local coordinate t. We are identifying c with 0. Recall that the remark after Proposition 2.4.9 gives us a local description of the germ of every singular fiber with respect to the j-map (assuming that it is not identically 0 or 1). We are going to collect these *normal forms* in the table below, for ease of reference.

All parts of the table are already quite clear; for instance, let us explain the I_0 row, as the others can be obtained analogously. From the a, b, δ table 2.4, we know that for each value of $j(0)$, the multiplicity m must satisfy some congruence relations. In particular, if $j(0) = 0$, $m = 3a$ must be multiple of 3 (or ∞); if $j(0) = 1$, $m = 2b$ must be multiple of 2 (or ∞); if $j(0) \neq 0, 1, \infty$, m can be any positive number. Therefore we get 6 different cases: when $j(0) \neq 0, 1, \infty$, we use the explicit formulas from Proposition 2.4.9; when $j(0) = 0, 1$, the a, b, δ table 2.4 already gives us (locally analitically) the coefficients $A(t)$, $B(t)$.

Fiber	$j(t)$	$A(t)$	$B(t)$
I_0	0	0	1
	1	1	0
	$j \neq 0, 1$	$-3j(j-1)$	$2j(j-1)^2$
	t^{3n}	t^n	1
	$1+t^{2n}$	1	t^n
	$j+t^n,\ j \neq 0,1$	$-3(j+t^n)(j+t^n-1)$	$2(j+t^n)(j+t^n-1)^2$
I_n	t^{-n}	$-3(1-t^n)$	$2(1-t^n)^2$
I_0^*	0	0	t^3
	1	t^2	0
	$j \neq 0, 1$	$-3j(j-1)t^2$	$2j(j-1)^2t^3$
	t^{3n}	t^{n+2}	t^3
	$1+t^{2n}$	t^2	t^{n+3}
	$j+t^n,\ j \neq 0,1$	$-3t^2(j+t^n)(j+t^n-1)$	$2t^3(j+t^n)(j+t^n-1)^2$
I_n^*	t^{-n}	$-3t^2(1-t^n)$	$2t^3(1-t^n)^2$
II	0	0	t
	t^{3n+1}	t^{n+1}	t
III	1	t	0
	$1+t^{2n+1}$	t	t^{n+2}
IV	0	0	t^2
	t^{3n+2}	t^{n+2}	t^2
IV^*	0	0	t^4
	t^{3n+1}	t^{n+3}	t^4
III^*	1	t^3	0
	$1+t^{2n+1}$	t^3	t^{n+5}
II^*	0	0	t^5
	t^{3n+2}	t^{n+4}	t^5

Table 2.7: Normal forms for germs of singular fibers. The fibration is locally written as $y^2 = x^3 + A(t)x + B(t)$.

Let $\pi\colon X \to \Delta$ be (the restriction of) a Weierstrass fibration, with a unique singular fiber X_0 over $0 \in \Delta$. For any $0 \neq p \in \Delta$, the first cohomology group $H^1(X_p, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Fix a point $p \in C$ away from 0, and consider a loop $\gamma \subseteq \Delta \backslash \{0\}$ that winds around 0 once, in the counterclockwise direction, parametrized by the interval $[0,1]$ such that $\gamma(0) = \gamma(1) = p$. For every $s \in [0,1]$, the fiber $X_{\gamma(s)}$ is smooth, hence we can identify the groups $H^1(X_{\gamma(s)}, \mathbb{Z})$ with the starting group $H^1(X_p, \mathbb{Z})$. Once we return on the point p, i.e. at $s = 1$, we obtain a new pair of generators for the free abelian group $H^1(X_p, \mathbb{Z})$, and thus an automorphism of $H^1(X_p, \mathbb{Z})$. It is immediate to see that the starting point p and the loop γ do not affect this automorphism: more precisely, it doesn't change if we change γ with a homotopy equivalent loop. Therefore, the element $\mathscr{M}_0 \in \mathrm{Aut}(H^1(X_p, \mathbb{Z}))$ depends only on the germ of the fiber X_0, and we call it the *local monodromy* around 0. If we fix a basis for the cohomology group $H^1(X_p, \mathbb{Z})$, the monodromy $\mathscr{M}_0$ can be interpreted as an invertible matrix in $\mathrm{GL}(2, \mathbb{Z})$. But since the fiber X_p is a smooth elliptic curve, we can find a lattice $\Lambda_p = \langle \tau_1(p), \tau_2(p) \rangle_{\mathbb{Z}}$ of $\mathbb{C}$ such that $X_p \cong \mathbb{C}/\Lambda_p$. In this way, we can identify the pair $\tau_1(p), \tau_2(p)$ with a basis for the group $H^1(X_p, \mathbb{Z})$, and compute the monodromy seeing how these two generators move after circling the origin. For instance, we can embed $\mathbb{Z} \oplus \mathbb{Z} = H^1(X_{\gamma(s)}, \mathbb{Z}) \subseteq H^1(X_{\gamma(s)}, \mathcal{O}_{X_{\gamma(s)}}) = \mathbb{C}$, and the monodromy $\mathscr{M}_0$ extends to a holomorphic automorphism of $H^1(X_p, \mathcal{O}_{X_{\gamma(s)}})$; therefore $\det(\mathscr{M}_0) = 1$, and we have obtained that the local monodromy around 0 is a matrix in $\mathrm{SL}(2, \mathbb{Z})$. Every time we will have a germ of a Weierstrass fibration, we will tacitly fix the basis for $H^1(X_t, \mathbb{Z})$ given by the generators of the lattice of periods Λ_t for each $0 \neq t \in \Delta$.

Since the monodromy depends only on the germ of the fiber X_0, we can limit ourselves to the computation of the matrix $\mathscr{M}_0$ in the cases discussed above, in Table 2.7. Now we are going to see how to limit the explicit computations as much as possible.

Take as usual a Weierstrass fibration $\pi\colon X \to \Delta$ with only singular fiber X_0 over $0 \in \Delta$, and base change the base disc with the m-cyclic covering ramified over 0. In other words, consider the commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ \Delta \arrow[r, "\varphi_m"'] & \Delta \end{tikzcd}$$

where s, t are local coordinates for Δ such that $t = \varphi_m(s) = s^m$. The germ of the fiber X'_0 depends only on the germ of the fiber X_0 and m, hence we can use Table 2.7 to understand how the singular fiber changes after the base change.

Proposition 2.5.3. *The singular fibers change as in the following table under base change:*

Fiber	m	**Fiber after the base change**
I_0	any $m \geq 1$	I_0
I_n	any $m \geq 1$	I_{mn}
I_n^*	0 (mod 2)	I_{mn}
	1 (mod 2)	I_{mn}^*
II	0 (mod 6)	I_0
	1 (mod 6)	II
	2 (mod 6)	IV
	3 (mod 6)	I_0^*
	4 (mod 6)	IV^*
	5 (mod 6)	II^*
III	0 (mod 4)	I_0
	1 (mod 4)	III
	2 (mod 4)	I_0^*
	3 (mod 4)	III^*
IV	0 (mod 3)	I_0
	1 (mod 3)	IV
	2 (mod 3)	IV^*
II^*	0 (mod 6)	I_0
	1 (mod 6)	II^*
	2 (mod 6)	IV^*
	3 (mod 6)	I_0^*
	4 (mod 6)	IV
	5 (mod 6)	II
III^*	0 (mod 4)	I_0
	1 (mod 4)	III^*
	2 (mod 4)	I_0^*
	3 (mod 4)	III
IV^*	0 (mod 3)	I_0
	1 (mod 3)	IV^*
	2 (mod 3)	IV

Table 2.8: Effect of base changes on singular fibers.

Proof. Everything boils down to a boring case by case check. We will explain some of them, and the remaining should be done in the exact same way. Take for instance an I_n^* fiber, and base change it with $t = s^m$. We obtain a new singular fiber, whose j-map has a pole at 0 of order mn, hence it can only be I_{mn} or I_{mn}^*. So focus on the coefficients $A(t)$, $B(t)$: they trasform into $A'(s) = -3s^{2m}(1-s^{mn})$ and $B'(s) = 2s^{3m}(1 - s^{mn})^2$, and after the necessary normalization (dividing A' by s^4 and B' by s^6 until $\mu_0(A') \leq 3$ or $\mu_0(B') \leq 5$) we obtain an I_{mn} fiber if $m \equiv 0 \pmod 2$, and an I_{mn}^* fiber if $m \equiv 1 \pmod 2$.
Take instead a II fiber. The new j-map acquires a zero of order $m(3n + 1)$ at 0, and A, B become $A'(s) = s^{m(n+1)}$, $B'(s) = s^m$. After the usual normalization, we see for instance that if $m \equiv 0 \pmod 6$, $B'(s) = 1$, hence the new fiber is smooth. If $m \equiv 1 \pmod 6$, $B'(s) = s$, hence the new fiber remains of type II. If $m \equiv 2 \pmod 6$, $B'(s) = s^2$, hence the new fiber is of type IV, and so on. □

The I_n fibers are said *semistable*, and the name comes from the geometric invariant theory. For the same reason, we will call the smooth fibers *stable*, while the others *unstable*. In particular, the fibers I_0^*, II, III, IV, IV^*, III^*, II^* are said to have *potential stable reduction*: they become smooth after a base change of finite order.

How is this related to the study of the local monodromy? The only remark to make is that the unit circle $\gamma = e^{2\pi i\theta}$ on Δ is carried into the same circle, repeated m times, by the covering φ_m; in other words, the local monodromy $\mathscr{M}_0'$ around X_0' coincides with the m-fold monodromy $\mathscr{M}_0^m$ around X_0. Since these monodromies are nothing more than matrices in $\mathrm{GL}(2, \mathbb{Z})$, it is quite easy to determine them in each case.

Obviously, if X_0 is smooth, the monodromy $\mathscr{M}_0$ is trivial, i.e. it is the identity: the smoothness of the central fiber gives that the loop γ can be contracted to the point 0, contributing to a trivial action on $H^1(X_0, \mathbb{Z})$. The first non-trivial monodromy we want to compute is relative to an I_1 fiber. Since by Table 2.7 the germ of an I_1 fiber is unique, we can choose any local description. Following [Kod60], we consider

$$X = \{([x, y, z], s) \in \mathbb{P}^2 \times \Delta \mid y^2z = 4x^3 + (s-3)xz^2 + (s-1)z^3\};$$

it is immediate to check that the central fiber X_0 is a nodal rational curve with node at $[-\frac{1}{2}, 0, 1]$, and all the other fibers are smooth.
For $s \in \mathbb{R}$ and $0 < s < 1$, the polynomial $4x^3 + (s-3)x + (s-1)$ has three real distinct roots $s_1 < s_2 < s_3$; when $s \to 0$, the first two tend to $-\frac{1}{2}$, while s_3 tends to 1. Projecting the fiber X_s onto the x, z variables, we obtain a double covering $\phi_s \colon X_s \to \mathbb{P}^1$ branched at s_1, s_2, s_3 and ∞. The part of X_s over the real axis consists of four circles: the minimum and maximum equators b, b' and two symmetric meridians a, a' of the complex torus X_s (we choose a such that its image on the real axis is $(-\infty, s_1)$). When $s \to 0$, $s_2 \to s_1$ and the minimum equator b contracts to a point (the nodal point of X_0). Now we need a theorem of the so-called *Picard-Lefschetz theory*; for a detailed discussion, we refer to [Lam81]. In this setting b is said to be a *vanishing cycle*, i.e. it contracts to a point when approaching the singular fiber, and it can be used to compute the monodromy around X_0:

Theorem 2.5.4. *If $\alpha \in H^1(X_s, \mathbb{Z})$, and $\mathscr{M}_0$ is the monodromy around the unique singular fiber X_0, then*

$$\mathscr{M}_0(\alpha) = \alpha - \sum_i (\alpha \cdot e_i)e_i,$$

where the e_i are the vanishing cycles and $\alpha \cdot e_i$ indicates the intersection product of α and e_i.

Since a and b (actually, their Poincarè duals) generate the cohomology group $H^1(X_s, \mathbb{Z})$, and the intersection product of a and b is -1, we have $\mathscr{M}_0(a) = a + b$ and $\mathscr{M}_0(b) = b$. In other words,

$$\mathscr{M}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } I_1.$$

Notice that we are writing the matrix $\mathscr{M}_0$ with the following convention: we identify a with $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and b with $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, and we take $\mathscr{M}_0$ such that the transpose ${}^t\mathscr{M}_0$ acts on the vectors $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ as the monodromy acts on a and b. The reason for this convention will be explained later.

From this explicit computation, it is immediate to deduce the monodromy around an I_n fiber: using Table 2.8 and the remark above, the monodromy is

$$\mathscr{M}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \text{ for } I_n.$$

Remark 2.5.5. The I_n germ, for $n \geq 1$, can be constructed as $(\mathbb{C} \times \Delta)/(\frac{n}{2\pi i}\mathbb{Z}\log(s) + \mathbb{Z})$ (see [BPV84, Section III.15]). By using the exponential $\exp(\bullet) = e^{2\pi i \bullet}$ to factor out the second $\mathbb{Z}$, we see that we can write the fibration as $(\mathbb{C}^* \times \Delta)/\{t^{nj} \mid j \in \mathbb{Z}\}$. This is the *Jacobi form* of the fibration with central fiber I_n.

Before computing the other monodromies, we want to point out an alternative representation of an elliptic surface with smooth fibers, the *Jacobian fibration*; let $\pi\colon X \to \Delta$ be a Weierstrass fibration with only smooth fibers. The idea is quite natural: each fiber X_s is a smooth elliptic curve, so it is isomorphic to some torus $\mathbb{C}/(\mathbb{Z}\tau(s) + \mathbb{Z})$, with $\tau(s) \in \mathbb{C}$ in the upper half plane. τ is a holomorphic function, therefore the projection

$$\pi_2\colon (\mathbb{C} \times \Delta)/(\mathbb{Z}\tau(s) + \mathbb{Z}) \longrightarrow \Delta$$

on the second factor defines an elliptic fibration isomorphic to our given one: the isomorphism is simply constructed by patching together all the pointwise isomorphisms.

Recall that we have a standard action of $\mathrm{SL}(2, \mathbb{Z})$ on the upper half plane, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Therefore, we would like to define an action of $\mathrm{SL}(2, \mathbb{Z})$ on the basis $\{\tau_2, \tau_1\}$ of $H^1(X_s, \mathbb{Z})$ that corresponds to the previous action when we normalize the basis $\{\tau_2, \tau_1\}$ to $\left\{\tau = \frac{\tau_2}{\tau_1}, 1\right\}$. For, identify τ_2 with $\binom{1}{0}$ and τ_1 with $\binom{0}{1}$, and define the action of $\mathrm{SL}(2, \mathbb{Z})$ on $H^1(X_s, \mathbb{Z})$ that acts on τ_2, τ_1 by matrix-vector multiplication. Notice that the first action above is a *left* action, while the one just defined is a *right action*; hence we change our last definition, and we define that $A \in \mathrm{SL}(2, \mathbb{Z})$ acts on τ_2, τ_1 as its transpose ${}^tA$. Now both the actions are *left* actions, and we can easily check that they are compatible, in the sense that

$$A \cdot \{\tau, 1\} = \{A \cdot \tau, 1\}$$

after the usual normalization. This is the reason for the strange convention we introduced when computing the monodromies around the I_n fibers.

Remark 2.5.6. The action of $\mathrm{SL}(2, \mathbb{Z})$ on τ only determines the monodromy up to sign!

The only disadvantage of this alternative representation is that it is not easy to determine the j-invariant of a certain smooth fiber X_s. However, we can deduce the multiplicity of j from the local behaviour of τ around the central fiber. Indeed, j is a modular form of weight 0 from the upper half plane to $\mathbb{C}$ such that

$$j(\tau) = \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2},$$

where g_2 and g_3 are modular forms of weight respectively 4 and 6 (we refer to [Ser73] for the definitions and the results). Now recall the *valence formula*, whose proof can again be found in [Ser73]:

Theorem 2.5.7 (Valence formula)**.** *Let f be a meromorphic modular form of weight k. Then*

$$\mu_\infty(f) + \frac{1}{3}\mu_{\zeta_3}(f) + \frac{1}{2}\mu_i(f) + \sum_a \mu_a(f) = \frac{k}{12},$$

where a varies among all the other points of the fundamental domain for $\mathrm{SL}(2, \mathbb{Z})$.

Consequently, we have:

Corollary 2.5.8. *Let $\tau(s) = \tau_0 + s^m$ be a holomorphic function defined over $\Delta\backslash\{0\}$. Then the j-map is given locally around* 0 *by $j(s) = j(\tau(s)) = j(\tau_0) + s^{\alpha m}$, where*

$$\alpha = \begin{cases} 3 & \textit{if } \tau_0 = \zeta_3 \\ 2 & \textit{if } \tau_0 = i \\ 1 & \textit{otherwise} \end{cases}$$

Proof. In any case, the denominator of the j function is non-zero. If $\tau_0 = \zeta_3$, then the local behaviour of j is decided by the local behaviour of g_2, which is a modular form of weight 4; by the valence formula, g_2 has a simple zero at ζ_3, hence j has a zero of multiplicity 3 at ζ_3. By composing the power series, we obtain that $j(\tau(s))$ looks locally like s^{3m}.
If instead $\tau_0 = i$, we proceed similarly: since $j(i) = 1$, the function $j(\tau) - 1 = \frac{27g_3(\tau)^2}{g_2(\tau)^3 - 27g_3(\tau)^2}$ has a non-zero denominator. Moreover, by the valence formula, g_3 has a simple pole at i, hence $j(\tau) - 1$ has a zero of multiplicity 2 at i. By composing the power series, we obtain that $j(\tau(s))$ looks locally like $1 + s^{2m}$.
The last point is obvious, since j is not branched outside 0 and 1. □

This explains why a Weierstrass fibration with smooth central fiber X_0 such that $j(X_0) = 0$ (respectively, $j(X_0) = 1$) must have a j-map with multiplicity multiple of 3 (respectively, 2).

Now we are ready to compute the remaining monodromies; we will see that they come quite easily from the monodromy of the I_n fibers (which in turn come immediately from the monodromy of the nodal fiber I_1).

Let's start with I_0^*. Consider a smooth Jacobian fibration given by $\pi\colon X = (\mathbb{C} \times \Delta)/(\mathbb{Z}\tau(s) + \mathbb{Z}) \to \Delta$, where $\tau(s) = \tau_0 + s^{2m}$. τ_0 and m are chosen according to the germ of I_0^* fiber we want to construct: just look at Table 2.7. The map $(c, s) \mapsto (-c, -s)$ defined on $\mathbb{C} \times \Delta$ induces an involution ι of X, and the quotient surface $Y = X/\langle\iota\rangle$ has a singular fiber over 0. By Hurwitz's formula, this central fiber is a rational double curve with 4 singularities (corresponding to the 4 fixed points of ι), and by Proposition 1.3.20 these 4 singularities are of type A_1. After resolving them, we see that we obtain an I_0^* fiber over 0. Consequently we have a commutative diagram ($\widetilde{Y}$ is the resolution of the singularities of Y)

$$\begin{tikzcd} X \arrow[r, dashed, "p"] \arrow[d, "\pi"'] & \widetilde{Y} \arrow[d, "\pi_Y"] \\ \Delta \arrow[r, "\varphi_2"'] & \Delta_Y \end{tikzcd}$$

(p is only defined outside the central fiber) where $t = \varphi_2(s) = s^2$ is a base change of order 2. In particular, the j-map for $\widetilde{Y}$ looks locally like $\tau_0 + t^m$; notice that with this construction we have built all the possible germs of I_0^* fibers.
If γ is a small loop (starting and ending at t) in Δ_Y around 0 that circles the origin (in the counterclockwise direction) once, then the lift of γ through φ_2 gives "half a loop" in Δ, i.e. an arc from s to $-s$; therefore the monodromy $\mathscr{M}_0$ around the I_0^* fiber in $\widetilde{Y}$ lifts to an isomorphism $H^1(X_s, \mathbb{Z}) \to H^1(X_{-s}, \mathbb{Z})$ that sends 1 to -1. In conclusion, we have that $\mathscr{M}_0$ is a matrix in $\mathrm{SL}(2, \mathbb{Z})$ such that

$$\mathscr{M}_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad {}^t\mathscr{M}_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

hence

$$\mathscr{M}_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for } I_0^*.$$

The argument for the I_n^* fibers is similar: we start with a Weierstrass fibration $\pi\colon X \to \Delta$ with a central I_{2n} fiber. On this fibration we consider the involution ι induced by the map $(x, y, t) \mapsto (x, -y, -t)$, where as usual X is given by $y^2 = x^3 + A(t)x + B(t)$. ι extends uniquely to the resolution of singularities $\widetilde{X}$ of X (see [BPV84, Proposition III.8.5]), and in particular ι acts on the cycle of $2n$ rational curves forming the central singular fiber $\widetilde{X}_0$. If C_1 if the irreducible component of $\widetilde{X}_0$ meeting the zero section S_0, and the other components are labeled in circle, we have that ι acts on C_1 and C_{n+1} separately, and identifies C_i with C_{2n+2-i} for $i = 2, \ldots, n$. Hence the quotient $Y = \widetilde{X}_0/\langle\iota\rangle$ has a central fiber consisting of $n-1$ smooth double rational curves (the images of the C_i and C_{2n+2-i}) and two double singular rational curves (attached at the begininng and at the end of the string of the previous $n-1$

curves), each with two singularities of type A_1. As above, the resolution of the singularities gives a fibration with a central I_n^* fiber, and we obtain a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, dashed, "p"] \arrow[d, "\pi"'] & \widetilde{Y} \arrow[d, "\pi_Y"] \\ \Delta \arrow[r, "\varphi_2"'] & \Delta_Y \end{tikzcd}$$

Analogous to the case $n = 0$, we have that the monodromy $\mathscr{M}_0$ around the I_n^* fiber is a matrix in $\mathrm{SL}(2,\mathbb{Z})$ such that

$$\mathscr{M}_0^2 = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad {}^t\mathscr{M}_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

hence

$$\mathscr{M}_0 = \begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix} \text{ for } I_n^*.$$

We remain with the 6 singular fibers II, III, IV, IV^*, III^* and II^*. Let's start with II^*.

We start with a smooth fibration $\pi\colon X = (\mathbb{C}\times\Delta)/(\mathbb{Z}\tau(s)+\mathbb{Z}) \to \Delta$, where

$$\tau(s) = \frac{\zeta_3 - \zeta_3^2 s^m}{1 - s^m}.$$

As in the I_0^* case, we let m vary in order to obtain all the possible germs: in this case we ask that $m \equiv 4 \pmod 6$. Now let the group $G = \mathbb{Z}/6\mathbb{Z}$ act on X with the action induced by

$$\mu\colon (c,s) \longmapsto \big((\tau(s)+1)^{-1}c, \zeta_6 s\big)$$

on $\mathbb{C}\times\Delta$, where μ corresponds to $\overline{1}$. Notice that μ acts on X_0 as multiplication by $(\zeta_3+1)^{-1} = \zeta_6^{-1}$. In the quotient $Y = X/G$, the central fiber becomes a rational curve of multiplicity 6, with some singular points coming from the fixed points of G. By Hurwitz's formula we have that the ramification gives a contribution equal to 12, and in particular

$$\sum_i \frac{6}{m_i}(m_i - 1) = 12,$$

where the m_i's are the multiplicities of the ramification points in X_0. The numbers $\frac{6}{m_i}(m_i-1)$ can only vary among $3,4,5$ (respectively, if $m_i = 2,3,6$), and since the origin of X_0 has multiplicity 6, the only possibility is that Y_0 has 3 singular points: P_1 is the image of the origin of X_0, P_2 is the image of the points identified by $2G$, and P_3 is the image of the points identified by $3G$. Since the action of μ is locally $(c,s) \mapsto (\zeta_{\frac{6}{k}}^{-1} c, \zeta_6 s)$ around P_k, $k = 1,2,3$, then by Proposition 1.3.20 we have that these 3 singularities are respectively of type A_5, A_2 and A_1. Therefore the resolution of the singularities $\widetilde{Y}$ has a central fiber $\widetilde{Y}_0$ composed by a rational curve of multiplicity 6, on which three separate strings of rational curves are attached, formed respectively by $5,2$ and 1 curves. Consequently the central fiber $\widetilde{Y}_0$ is a II^* singular fiber. Now the usual diagram

$$\begin{tikzcd} X \arrow[r, dashed, "p"] \arrow[d, "\pi"'] & \widetilde{Y} \arrow[d, "\pi_Y"] \\ \Delta \arrow[r, "\varphi_6"'] & \Delta_Y \end{tikzcd}$$

says that, since the j-map of X has the form $j(s) = s^{3m}$, the j-map around $\widetilde{Y}_0$ is $j(t) = t^{\frac{m}{2}}$, and by assumption $\frac{m}{2} \equiv 2 \pmod 3$. This shows that we have constructed all the germs of II^* fibers.
Now the monodromy can easily computed: $\tau(s)$ is transformed into

$$\tau(\zeta_6 s) = \frac{\zeta_3 - \zeta_3 s^m}{1 - \zeta_3^2 s^m} = -(\tau(s)+1)^{-1},$$

thus the monodromy $\mathscr{M}_0$ is one of the matrices

$$\pm\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}.$$

Since the matrix with the $-$ has order 3 in $\mathrm{SL}(2,\mathbb{Z})$, if this were the monodromy of the II^* fiber, then its third power (i.e., the identity) would be the monodromy of an I_0^* fiber, a contradiction. We conclude that

$$\mathscr{M}_0 = \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix} \text{ for } II^*.$$

This is coherent with our definition of the action of the monodromy: we identify the group $H^1(X_{\zeta_6 s},\mathbb{Z})$ with $H^1(X_s,\mathbb{Z})$ circling the origin, i.e. identifying the basis $\{\tau(\zeta_6 s), 1\}$ of $H^1(X_{\zeta_6 s},\mathbb{Z})$ with the basis

$$\{\tau(\zeta_6 s)(\tau(s)+1), 1\cdot(\tau(s)+1)\} = \{-1, \tau(s)+1\}$$

of $H^1(X_s,\mathbb{Z})$ (we have identified the bases via $\mu^{-1}\colon \mathbb{C}\times\{\zeta_6\}\to\mathbb{C}\times\{1\}$), hence the monodromy matrix is such that

$${}^t\mathscr{M}_0 = \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix},$$

as found above.

We deal with the III^* fibers similarly: we start with a smooth fibration $\pi\colon X = (\mathbb{C}\times\Delta)/(\mathbb{Z}\tau(s)+\mathbb{Z})\to\Delta$, where

$$\tau(s) = \frac{i+is^m}{1-s^m},$$

and $m\equiv 2 \pmod 4$. We let the group $G=\mathbb{Z}/4\mathbb{Z}$ act on X with the action induced by

$$\mu\colon (c,s)\longmapsto \big(\tau(s)^{-1}c, is\big)$$

on $\mathbb{C}\times\Delta$. μ acts on X_0 as the multiplication by $i^{-1}=-i$. As above, the central fiber becomes a rational curve of multiplicity 4 in the quotient $Y=X/G$, containing 3 singular points of multiplicities $4,4,2$. Thus Y_0 contains two singular points of type A_3 and one singular point of type A_1. Resolving these singularities, we see that Y_0 becomes a singular fiber of type III^* in $\widetilde{Y}$. Notice that by Corollary 2.5.8, the j-map of $\widetilde{Y}$ is locally $1+s^{\frac{m}{2}}$, and $\frac{m}{2}\equiv 1 \pmod 2$, as desired. The rotation by a square angle transforms $\tau(s)$ into

$$\tau(is) = \frac{i-is^m}{1+s^m} = -\tau(s)^{-1},$$

and since we identify the basis $\{\tau(is),1\}$ of $H^1(X_{is},\mathbb{Z})$ with the basis

$$\{\tau(is)\tau(s), 1\cdot\tau(s)\} = \{-1,\tau(s)\}$$

of $H^1(X_s,\mathbb{Z})$, the monodromy matrix is such that

$${}^t\mathscr{M}_0 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

hence

$$\mathscr{M}_0 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \text{ for } III^*.$$

We can compute the monodromies of the other 4 singular fibers base changing the fibrations just described, and using Table 2.8. Actually, we are being a little sloppy here: when we base change the two previous fibrations, we do not realize *all* the germs of II, III, IV and IV^* fibers, as we miss the ones with low multiplicity of the j-map. The solution to this is realizing all the remaining germs explicitly as above, and computing their monodromies. Since these constructions are quite similar to the ones just described, we omit them; the interested reader can consult [BPV84, Section V.10] to fill in the last details.

Theorem 2.5.9. *The monodromies around each singular fiber are given by*

Fiber	Monodromy	Fiber	Monodromy
I_n	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
I_n^*	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$	III^*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
II^*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	IV^*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

Table 2.9: Monodromy matrices around singular fibers.

In particular, the monodromy depends only on the type of singular fiber.

Recall that the two matrices $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ generate $\mathrm{SL}(2,\mathbb{Z})$: this will be useful in the following.

As usual, we would like to globalize this, in order to obtain an action around all singular fibers at once. We know that the map $j\colon \mathfrak{h} \to \mathbb{C}$ associating to a τ in the upper half-plane $\mathfrak{h}$ the j-invariant of the complex torus $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$, is a covering, branched over the two points $\{0,1\}$. Therefore the restriction $j\colon \mathfrak{h}\backslash j^{-1}\{0,1\} \to \mathbb{C}\backslash\{0,1\}$ is a topological covering, and the group $\mathbb{P}\,\mathrm{SL}(2,\mathbb{Z})$ acts transitively on each preimage; hence we have a natural map $\alpha\colon \pi_1(\mathbb{C}\backslash\{0,1\}) \to \mathbb{P}\,\mathrm{SL}(2,\mathbb{Z})$ associating to a loop γ the matrix in $\mathbb{P}\,\mathrm{SL}(2,\mathbb{Z})$ acting on the preimage $j^{-1}(\gamma(0))$ as the lift of the loop γ.

Now let $J\colon C \to \mathbb{P}^1$ be any non-costant map, and choose a finite subset $S \subseteq C$ such that the restriction of J to $C\backslash S$ is never $0, 1$ or ∞. Then $J\colon C\backslash S \to \mathbb{C}\backslash\{0,1\}$ is again non-costant, and induces a map $J_*\colon \pi_1(C\backslash S) \to \pi_1(\mathbb{C}\backslash\{0,1\})$; we can compose this homomorphism with α, and obtain another homomorphism

$$J_{\#}\colon \pi_1(C\backslash S) \longrightarrow \mathbb{P}\,\mathrm{SL}(2,\mathbb{Z}).$$

This construction is an attempt to generalize the local monodromy: if $C = \Delta$ is a small disc around 0 and $J\colon \Delta \to \mathbb{P}^1$ is the j-map of an elliptic surface over C, then $J_{\#}$ can be thought as the monodromy of the function $\tau(s)$ around 0. As we pointed out previously, this does not determine uniquely the monodromy around X_0, but only up to a sign. However, we have a commutative diagram

$$\begin{tikzcd} \pi_1(\Delta\backslash\{0\}) \arrow[r, "\mathscr{M}_0"] \arrow[dr, "J_{\#}"'] & \mathrm{SL}(2,\mathbb{Z}) \arrow[d] \\ & \mathbb{P}\,\mathrm{SL}(2,\mathbb{Z}) \end{tikzcd}$$

where the vertical arrow is simply the projection. Obviosly, the commutativity must be interpreted "up to conjugacy", if we do not fix a basis for the first cohomology group of a smooth fiber.

The local monodromy easily globalizes: if we choose S finite containing the images of all the singular fibers, we fix any base point $c \in C\backslash S$ and a basis for $H^1(X_c,\mathbb{Z})$, we have the *global monodromy*

$$G\colon \pi_1(C\backslash S) \longrightarrow \mathrm{SL}(2,\mathbb{Z});$$

this map is sometimes called the *homological invariant* of the elliptic surface X.

Definition 2.5.10. Let S be any finite subset of C, G a representation of $\pi_1(C\backslash S)$ into $\mathrm{SL}(2,\mathbb{Z})$, and

$J\colon C \to \mathbb{P}^1$ a non-costant map such that $S \supseteq J^{-1}\{0,1,\infty\}$. We say that G *belongs to* J if the diagram

$$\begin{tikzcd} \pi_1(C\backslash S) \arrow[r, "G"] \arrow[dr, "J_\#"'] & \mathrm{SL}(2,\mathbb{Z}) \arrow[d] \\ & \mathbb{P}\,\mathrm{SL}(2,\mathbb{Z}) \end{tikzcd}$$

commutes up to conjugacy.

The discussion above can be rephrased by saying that if $\pi\colon X \to C$ is an elliptic surface, and $S \subseteq C$ is a finite set such that π is smooth outside of S and $S \supseteq J^{-1}\{0,1,\infty\}$, then the homological invariant G of the elliptic surface belongs to J.
In other words, G is a lift of $J_\#$ from $\mathbb{P}\,\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{SL}(2,\mathbb{Z})$. Since $\pi_1(C\backslash S)$ is generated by $2g+|S|$ elements (the generators of $\pi_1(C)$ and the $|S|$ loops around the points of S), subject to one relation, and each matrix in $\mathbb{P}\,\mathrm{SL}(2,\mathbb{Z})$ can be lifted in two ways to $\mathrm{SL}(2,\mathbb{Z})$, we have:

Proposition 2.5.11. *Given a non-costant $J\colon C \to \mathbb{P}^1$, and a finite set S such that $S \supseteq J^{-1}\{0,1,\infty\}$, the number of homological invariants G belonging to J is $2^{2g+|S|-1}$.*

Recall that this number coincides with the number of elements in $\mathrm{Doub}_S(C)$ (see Proposition 2.4.7). This suggests that there should be a 1-1 corresponence between lifts of $J_\#$ and elements of the double cover group relative to S, i.e. between homological invariants belonging to a fixed J and elliptic surfaces with J as j-map. This is indeed the case, as the next result shows. With abuse of notation, we will denote also with G the locally costant sheaf, locally isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, over $C\backslash S$ induced by the homological invariant G: it is well defined, even though G is defined up to conjugacy, since the conjugation of G produces a sheaf isomorphic to G.

Proposition 2.5.12. *Let $J\colon C \to \mathbb{P}^1$ be any non-costant map, and $G\colon \pi_1(C\backslash S) \to \mathrm{SL}(2,\mathbb{Z})$ a representation belonging to J. Then there exists a unique Weierstrass fibration $X(J,G)$ with J as j-map and G as homological invariant.*

Proof. The local existence follows from Proposition 2.4.9 and the fact that all possible local monodromies are realizable; the local uniqueness is given by the uniqueness of the germ of the fiber and by the fact that the monodromy identifies the type of singular fiber. Thus we only have to patch together these local descriptions.
Let $\{U_i\}$ be a cover of sufficiently small analytic discs, such that each $s \in S$ is contained in only one U_i, and every intersection $U_i \cap U_j$ is again an analytic disc; by the first part of the proof, there exist unique elliptic surfaces $\pi_i\colon X_i \to U_i$ with section σ_i, $J|_{U_i}$ as j-map and $G|_{U_i}$ as local monodromy; considering G as a sheaf, this gives an identification α_i of $R^1(\pi_i)_*\mathbb{Z}$ with $G|_{U_i}$.
By our assumptions on the cover $\{U_i\}$, the fibrations π_i and π_j have the same j-map over $U_i \cap U_j$, and trivial monodromy, so there exists an isomorphism $\varphi_{ij}\colon X_j \to X_i$ respecting the fibrations π_i, π_j, mapping the section σ_j to σ_i, and respecting the previous identifications, i.e. $\alpha_i \circ (\varphi_{ij})_* = \alpha_j$. Notice that these isomorphisms φ_{ij} are all unique: the non-trivial automorphisms of X_i preserving the fibration and the section must act non-trivially on the sheaf $R^1(\pi_i)_*\mathbb{Z}$.
Now, fixed any three indices i,j,k, the composition $\varphi_{ij}\circ\varphi_{jk}\circ\varphi_{ki}$ is an automorphism of $X_i|_{U_i\cap U_j\cap U_k}$ preserving the fibration, the section and the sheaf $R^1(\pi_i)_*\mathbb{Z}$, hence it is the identity. This shows that we can patch the π_i together via the φ_{ij}, giving an elliptic surface with section $\pi\colon X(J,G) \to C$. The uniqueness follows from the local uniqueness and the uniqueness of the isomorphisms φ_{ij}. □

Actually we have proved more: since the relation $\varphi_{ij}\circ\varphi_{jk}\circ\varphi_{ki} = \mathrm{id}$ is the cocycle relation, we have that the set of elliptic surfaces with J as j-map and G as homological invariant is in 1-1 correspondence with the set $H^1(C,\mathscr{F})$, where $\mathscr{F}$ is the sheaf of local sections of $X(J,G)$.

This explains why the monodromy of a singular fiber resulting after the transfer of $*$ process is opposite in sign with respect to the original monodromy: the transfer of $*$ process is locally the only non-trivial quadratic twist, with S composed by the only point under the singular fiber, and by the discussion above this corresponds to take the other lift of $J_{\#}$ from $\mathbb{P}\,\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{SL}(2,\mathbb{Z})$.

2.6 The Mordell-Weil Group of Sections

In this last section of theory, we fully exploit the assumption that the elliptic surfaces we work with admit a section. As usual, we will denote by $\pi\colon X \to C$ a smooth minimal elliptic surface with section S_0. First we want to study the Néron-Severi of the surface X; recall that the Néron-Severi group $\mathrm{NS}(X)$ is defined so that it fits into the short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0,$$

where $\mathrm{Pic}^0(X) = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}$ has by Hodge theory a natural structure of complex torus of dimension $q = q(X)$.

Lemma 2.6.1. *The pull-back map $\pi^*\colon \mathrm{Pic}(C) \to \mathrm{Pic}(X)$ is injective. Moreover, if X is not a product, i.e. $\mathscr{L} \neq \mathcal{O}_C$, then the restriction $\pi^*\colon \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(X)$ is an isomorphism.*

Proof. The first assertion follows from the projection formula: if $L \in \mathrm{Pic}(C)$ is any line bundle, then

$$\pi_*\pi^*L = \pi_*(\mathcal{O}_X \otimes \pi^*L) = \pi_*\mathcal{O}_X \otimes L = L.$$

The second assertion is equally easy: by Proposition 2.3.1 we have that $q = g$, therefore the restriction $\pi^*\colon \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(X)$ is an injective homomorphism between complex tori of the same dimension, and thus an isomorphism. □

Proposition 2.6.2. *Assume that the fundamental line bundle $\mathscr{L}$ has positive degree. Then the Néron-Severi group $\mathrm{NS}(X) = \frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(X)}$ is torsion-free. In particular, any torsion class in $\mathrm{Pic}(X)$ is the pull-back of a torsion class in $\mathrm{Pic}(C)$.*

Proof. Let $0 \neq T \in \mathrm{Pic}(X)$ be a torsion class. T has degree zero, so $H^0(X,T) = 0$ and therefore

$$h^2(X,T) = \chi(T) + h^1(X,T) \geq \chi(T) = \chi(\mathcal{O}_X) + \frac{T(K_X - T)}{2} = \chi(\mathcal{O}_X) = \deg(\mathscr{L}) \geq 1,$$

since $TD = 0$ for every divisor D (a multiple of T is linearly equivalent to the zero divisor). By Serre duality, we obtain an effective divisor $D \in |K_X - T|$. Now K_X is the pull-back of a line bundle in $\mathrm{Pic}(C)$, so, if F is a fiber of π, $DF = 0$, and D must be vertical. Moreover $D^2 = 0$, thus D must be a combination of fibers, since irreducible components of reducible fibers have negative-definite intersection form. By difference, T is also a combination of fibers, and therefore T is pulled back from a torsion class T' on C. But on C we have the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and therefore $T' \in \mathrm{Pic}^0(C)$. By the previous lemma, we have that $T = \pi^*T' \in \mathrm{Pic}^0(X)$. □

Remark 2.6.3. This depends heavily on the fact that we are assuming the existence of a section, and thus that there are no multiple fibers.

Corollary 2.6.4. *Assume again that $\deg(\mathscr{L}) \geq 1$. Then the quotient $\frac{\mathrm{Pic}(X)}{\mathrm{Pic}(C)}$ is isomorphic to the quotient $\frac{\mathrm{NS}(X)}{\mathbb{Z}F}$, where F is a fiber of π.*

Proof. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^0(C) \arrow[r] \arrow[d, equal] & \mathrm{Pic}(C) \arrow[r] \arrow[d, hook, "\pi^*"'] & \mathbb{Z} \arrow[r] \arrow[d, hook, "\alpha"'] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & \mathrm{NS}(X) \arrow[r] & 0
\end{tikzcd}$$

is sufficient to conclude, noticing that the image of α is precisely $\mathbb{Z}F$. □

The previous results efficiently relate the Picard groups of X and C. The other standard relation is the one between the Picard groups of X and the generic fiber X_η. First of all, consider the restriction map

$$\begin{array}{rccc} r\colon & \mathrm{Div}(X) & \longrightarrow & \mathrm{Div}(X_\eta) \\ & D & \longmapsto & D_\eta \end{array}$$

It is clearly surjective, since if $E \in \mathrm{Div}(X_\eta)$, the closure of E in X gives a preimage of E. Moreover, every vertical divisor on X has trivial image in $\mathrm{Div}(X_\eta)$, and every non-trivial irreducible curve on X has non-trivial image in $\mathrm{Div}(X_\eta)$, since it does intersect X_η. Clearly any element in $\mathrm{Div}(X)$ can be decomposed as the sum of a vertical and a horizontal divisor (in a unique way), thus we obtain an isomorphism

$$r\colon \frac{\mathrm{Div}(X)}{\mathrm{VDiv}(X)} \xrightarrow{\sim} \mathrm{Div}(X_\eta),$$

where $\mathrm{VDiv}(X)$ is the subgroup of vertical divisors on X. In order to relate $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X_\eta)$, we have to understand what happens at the level of principal divisors $\mathrm{PDiv}(X)$. The same restriction map gives an injective map

$$r\colon \frac{\mathrm{PDiv}(X)}{\mathrm{PDiv}(X) \cap \mathrm{VDiv}(X)} \hookrightarrow \mathrm{PDiv}(X_\eta);$$

again, r is surjective, since the function fields $\mathbb{C}(X)$ and $\mathbb{C}(X_\eta)$ coincide, i.e. every meromorphic function on X_η can be uniquely extended to a meromorphic function of X. In the end, we obtain an isomorphism

$$r\colon \frac{\mathrm{Pic}(X)}{\mathrm{VPic}(X)} \xrightarrow{\sim} \mathrm{Pic}(X_\eta),$$

where obviously $\mathrm{VPic}(X) = \{\mathcal{O}_X(V) \mid V \in \mathrm{VDiv}(X)\}$. Clearly $\pi^* \mathrm{Pic}(C) \subseteq \mathrm{VPic}(X)$, so the last isomorphism can equivalently be written as an isomorphism

$$r\colon \frac{\mathrm{NS}(X)}{\mathrm{NS}(X) \cap \mathrm{VPic}(X)} \xrightarrow{\sim} \mathrm{Pic}(X_\eta).$$

Now we turn to the main character of the section.

Definition 2.6.5. The *Mordell-Weil group of sections* of the elliptic surface $\pi\colon X \to C$, indicated as $\mathrm{MW}(X)$, is the set of sections of π, with group law defined fiber by fiber and as neutral element the given section S_0. More precisely, the sum is defined fiber by fiber on smooth fibers, and then we take the closure inside X.

The group law on $\mathrm{MW}(X)$ can be equivalently defined as the sum in the general fiber X_η: there is a 1-1 correspondence between $\mathrm{MW}(X)$ and rational points on X_η, given by the obvious restriction map and conversely by taking the closure of the point in X. Then, given two sections S_1, S_2 of π, they can be seen as divisors on X, and we can sum their images $(S_1)_\eta + (S_2)_\eta$ inside X_η. By our previous discussion, there exists a unique section $S_1 + S_2$ such that

$$(S_1 + S_2)_\eta = (S_1)_\eta + (S_2)_\eta, \quad \text{i.e.} \quad (S_1 + S_2)_\eta \sim (S_1)_\eta + (S_2)_\eta - p_0,$$

where $p_0 = (S_0)_\eta$ is the origin of the group law in X_η.

Now take any divisor $E \in \mathrm{Div}(X_\eta)$ and consider the sum $\sigma(E)$ (inside X_η) of the points of E; by Abel's Theorem we know that the summation map σ is trivial on principal divisors (giving an isomorphism between X_η and $\mathrm{Pic}^0(X_\eta)$), so we can quotient them out and obtain $\sigma\colon \mathrm{Pic}(X_\eta) \to \mathrm{MW}(X)$. Composing this with the surjective restriction map $r\colon \mathrm{NS}(X) \to \mathrm{Pic}(X_\eta)$, we get a map

$$\psi\colon \mathrm{NS}(X) \longrightarrow \mathrm{MW}(X)$$

that sends the class of a divisor $D \in \mathrm{Div}(X)$ into the closure of the point $\sigma(D_\eta)$. The discussion above completely characterizes the image and the kernel of the map ψ: let $A \subseteq \mathrm{NS}(X)$ be the subgroup $\langle \mathrm{VPic}(X), S_0 \rangle_{\mathbb{Z}}$ generated by the vertical divisors and the zero section.

Proposition 2.6.6. *There is an exact sequence*

$$0 \longrightarrow A \overset{i}{\hookrightarrow} \mathrm{NS}(X) \overset{\psi}{\longrightarrow} \mathrm{MW}(X) \longrightarrow 0.$$

Proof. The surjectivity of ψ is easy: any section $S \in \mathrm{MW}(X)$ can be interpreted as an horizontal divisor $S \in \mathrm{NS}(X)$, and the composition $\psi = \sigma \circ r$ simply restricts S to X_η and takes its closure in X, giving back S. Moreover, A is mapped into 0 by ψ: this is obvious for the zero section, and the vertical divisors do not intersect X_η.
Conversely, let D be any element in $\mathrm{Ker}(\psi)$, i.e. such that $\sigma(D_\eta) = p_0$. The difference $D_\eta - \deg(D_\eta)p_0$ is a divisor of degree 0 on X_η, with sum $\sigma(D_\eta - \deg(D_\eta)p_0) = p_0$, and so again by Abel's Theorem $D_\eta - \deg(D_\eta)p_0$ is linearly equivalent to the zero divisor on X_η. Since $\deg(D_\eta)$ is the intersection number DF of D with a fiber F of π, we can equivalently say that the divisor class $D - (DF)S_0 \in \mathrm{Pic}(X)$ restricts to 0 on X_η; therefore the isomorphism $\frac{\mathrm{Pic}(X)}{\mathrm{VPic}(X)} \cong \mathrm{Pic}(X_\eta)$ gives that $D - (DF)S_0$ is linearly equivalent to a vertical divisor V on X, i.e. $D = (DF)S_0 + V \in A$ as elements of $\mathrm{NS}(X)$. □

An immediate consequence is:

Corollary 2.6.7. *The Mordell-Weil group* $\mathrm{MW}(X)$ *is a finitely generated abelian group.*

Now let R be the sublattice of A generated by the vertical components not meeting the zero section S_0. Just by looking at Table 2.3, we see that R is a (finite) direct sum of lattices of types A_n, D_n, E_6, E_7, E_8. Clearly the rank of the lattice R is the sum of all the "local" ranks r_c; moreover r_c coincides with the number of irreducible components of the singular fiber X_c not meeting S_0, so it can be easily computed (it is the r column in the a, b, δ table 2.4).

If $U = \langle S_0, F\rangle$ is the rank 2 unimodular sublattice of A generated by the zero section S_0 and the fiber F (it is unimodular since the intersection matrix has the form $\left(\begin{smallmatrix} * & 1 \\ 1 & 0 \end{smallmatrix}\right)$), then obviously $A = U \oplus R$, and moreover R is the perpendicular space to U (in A), thus

$$A = U \overset{\perp}{\oplus} R.$$

In particular $\mathrm{rk}(A) = 2 + \mathrm{rk}(R)$, and combining Proposition 2.6.6 with the above remark, we obtain the so-called *Shioda-Tate formula*:

Corollary 2.6.8 (Shioda-Tate formula). *Let ρ be the Picard number of X, i.e. the rank of the Néron-Severi group* $\mathrm{NS}(X)$. *Then*

$$\rho = 2 + \sum_{c \in S} r_c + \mathrm{rk}(\mathrm{MW}(X)),$$

where S is the set of points of C corresponding to the singular fibers.

The above discussion also leads to another simple consequence: if we denote by $U^\perp$ the perpendicular space to U in $\mathrm{NS}(X)$, then $\mathrm{NS}(X) = U \oplus U^\perp$. This is because U is unimodular: we have that $U \cong U^*$, with $S_0^* = F$ and $F^* = S_0 - (S_0^2)F$ (we denote by F^* the element of U such that $F^*F = 1$ and $F^*S_0 = 0$, and analogous for S_0^*), therefore the map

$$\begin{array}{ccc} \mathrm{NS}(X) & \longrightarrow & U \oplus U^\perp \\ D & \longmapsto & ((DF^*)F^* + (DS_0^*)S_0^*, D - (DF^*)F^* - (DS_0^*)S_0^*) \end{array}$$

is the inverse of the obvious inclusion $U \oplus U^\perp \to \mathrm{NS}(X)$ induced by the sum. Thus we can reformulate Proposition 2.6.6:

Proposition 2.6.9. *There exists an exact sequence*

$$0 \longrightarrow R \overset{i}{\hookrightarrow} U^\perp \overset{\psi}{\longrightarrow} \mathrm{MW}(X) \longrightarrow 0.$$

Proof. Since ψ sends U to 0, we can eliminate the U term from A and $\mathrm{NS}(X)$ in the exact sequence of Proposition 2.6.6. □

Now consider the map $\beta\colon \mathrm{NS}(X) \to R^* = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Z})$ given by the intersection form on $\mathrm{NS}(X)$: if $D \in \mathrm{NS}(X)$, then the image $\beta(D)$ is the linear functional on R such that $\beta(D)(D') = DD'$ for every $D' \in R$. Recall that in Section 1.4 we have identified R^* with the group

$$R^{\#} = \{x \in R_{\mathbb{Q}} \mid \langle x, r\rangle \in \mathbb{Z}\ \forall r \in R\},$$

hence we can compose β with the projection $R^{\#} \to G_R = R^{\#}/R$, obtaining a well defined homomorphism

$$\gamma\colon \mathrm{MW}(X) \longrightarrow G_R,$$

since R goes to 0 under the projection, and U is mapped into 0 by β, because it is spanned by the components S_0, F not intersecting any of the generators of R.

Recall that in Section 1.4 we have completely determined the groups G_L, whenever L is an A_n, D_n, E_6, E_7 or E_8 lattice. Therefore consider the splitting

$$R = \bigoplus_{c \in S'}^{\perp} R_c,$$

where S' is the set of points of C with reducible fiber (i.e. those $c \in C$ such that $r_c > 0$), and R_c is the lattice of components of X_c not meeting the zero section S_0. The splitting is clearly orthogonal, and it induces another splitting

$$G_R = \bigoplus_{c \in S'}^{\perp} G_{R_c}.$$

Thanks to this decomposition, we have reduced our problem to investigate the whole G_R to the much simpler task to determine the "local" G_{R_c}'s. We have already done this: combining the results of Table 2.3 and Table 1.6 we obtain the following table:

X_c	G_{R_c}
I_n, $n \geq 2$	$\mathbb{Z}/n\mathbb{Z}$
I_n^*, n even	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
I_n^*, n odd	$\mathbb{Z}/4\mathbb{Z}$
IV, IV^*	$\mathbb{Z}/3\mathbb{Z}$
III, III^*	$\mathbb{Z}/2\mathbb{Z}$
II, II^*	$\{0\}$

Table 2.10: Local G_{R_c}'s for the reducible singular fibers.

The non-zero elements of G_{R_c} coincide with the cosets modulo R_c of the functionals $e_1^*, \ldots, e_k^* \in R_c^*$ such that $e_i^*(e_j) = \delta_{ij}$ for every $1 \leq i \leq k$, $1 \leq j \leq n$, where $e_1, \ldots, e_k$ are the multiplicity 1 components inside X_c, and $e_{k+1}, \ldots, e_n$ are all the other components: this can be easily checked case by case, since we have an explicit description of the groups G_{R_c}. For instance, if $R_c = A_{n-1}$, then the element $k \pmod n \in \mathbb{Z}/n\mathbb{Z} = G_{R_c}$ is the coset of the functional $e_k^* \in R_c^*$ that is 1 on the k^{th} component of the I_n fiber, and 0 elsewhere.

Consequently, the cardinality of G_{R_c} is exactly the number of multiplicity one components inside X_c (the last one being the one intersecting S_0): this number can be found in the d column of the a, b, δ table 2.4.

In the last part of the section, we would like to study the torsion inside $\mathrm{MW}(X)$. The first ingredient is the subgroup $\mathrm{MW}_0(X)$ of $\mathrm{MW}(X)$ defined as

$$\mathrm{MW}_0(X) = \{S \in \mathrm{MW}(X) \mid S \text{ and } S_0 \text{ meet the same component of } X_c\ \forall c \in C\}.$$

Clearly this is equivalent to asking that the sections S and S_0 meet the same component of X_c for all $c \in S'$. Actually we can say more: it is sufficient that S and S_0 meet the same component of X_c when X_c is a reducible fiber with $d > 1$, i.e. with at least two multiplicity one components.

Proposition 2.6.10. $\mathrm{MW}_0(X) = \mathrm{Ker}(\gamma)$.

Proof. By definition, R is the set of components not meeting S_0, thus not meeting any $S \in \mathrm{MW}_0(X)$; hence γ maps $\mathrm{MW}_0(X)$ into 0.
Conversely, if $S \notin \mathrm{MW}_0(X)$, then there exists a reducible fiber X_c such that S meets a multiplicity one component e of R_c. Therefore the projection of $\gamma(S)$ onto G_{R_c} is exactly e^*: S meets only e of all the components inside X_c. By our previous discussion, this projection of $\gamma(S)$ onto G_{R_c} is non-zero, hence $\gamma(S) \neq 0$. □

Using that G_R is a finite group, we have:

Corollary 2.6.11. $\mathrm{MW}_0(X)$ *has finite index in* $\mathrm{MW}(X)$.

We have introduced the subgroup $\mathrm{MW}_0(X)$ in order to study the torsion in $\mathrm{MW}(X)$; the next proposition motivates this:

Proposition 2.6.12. $\mathrm{MW}_0(X)$ *is torsion free if* $\deg(\mathscr{L}) \geq 1$.

Proof. Let $S \in \mathrm{MW}_0(X)$ be a torsion section of order $k \geq 2$. Then the divisor $kS - kS_0$ restricts to the zero divisor on the general fiber X_η, and thus $kS - kS_0$ is a vertical divisor V. However, since $S \in \mathrm{MW}_0(X)$, the difference $S - S_0$ cannot meet any vertical component; this forces $V = k(S - S_0)$ to meet no vertical components, and in particular $V^2 = 0$, i.e. V is a sum of fibers. If we projects these divisors onto $\mathrm{NS}(X)$, the fibers in V project onto the same element $F \in \mathrm{NS}(X)$, hence $kS - kS_0 = aF$ as elements of $\mathrm{NS}(X)$. If $s = SS_0$ and

$$l = -\deg(\mathscr{L}) = \deg(N_{S/X}) = \deg(\mathcal{O}_S(S)) = S^2 = S_0^2,$$

then intersecting the equality $kS - kS_0 = aF$ respectively with S and S_0 we get

$$kl - ks = a, \qquad ks - kl = a,$$

from which $a = 0$ and $s = l$. This is a contradiction, since $s \geq 0$ and $l = -\deg(\mathscr{L}) \leq -1$. □

Therefore, the torsion in $\mathrm{MW}(X)$ must be "contained" inside the finite group G_R; more precisely, if $\mathrm{TMW}(X)$ is the torsion subgroup of $\mathrm{MW}(X)$, we have:

Corollary 2.6.13. *If* $\deg(\mathscr{L}) \geq 1$*, then the restricted map* $\gamma\colon \mathrm{TMW}(X) \to G_R$ *is injective. In particular, a torsion section is completely determined by the vertical components it meets.*

Another interesting feature of torsion sections is that they cannot meet:

Proposition 2.6.14. *Let* S_1, S_2 *two distinct torsion sections. Then* S_1 *and* S_2 *are disjoint.*

Proof. Since $\mathrm{MW}(X)$ is a group, it suffices to assume that one of S_1 and S_2 is the zero section S_0. Let the other torsion section be any $S_0 \neq S \in \mathrm{TMW}(X)$.
We start with a simple remark: if we base change our elliptic surface, and S, S_0 do not meet after the base change, then they couldn't intersect before as well. If this were the case, S and S_0 would intersect at a point of a multiple component, and this is impossible since sections only meet multiplicity one components. Moreover, if a normalization is needed, we are only contracting some curve, hence

again, if they do not meet after the normalization, they couldn't intersect before. Therefore we can appropriately base change the elliptic surface and, working locally, we can assume that the fibration has a unique (semi)stable singular fiber, i.e. of type I_n, with $n \geq 0$.
If S and S_0 intersect at a point of a smooth fiber, then we can (locally) write the fibration as $\mathbb{C} \times \Delta/(\mathbb{Z}\tau(t) + \mathbb{Z})$, where t is a local coordinate on Δ, and τ is a holomorphic function. The zero section S_0 is obviously given by the identically zero map $z_0(t) \equiv 0$, where $z(t_0)$ is a coordinate on $\mathbb{C} \times \{t_0\}/(\mathbb{Z}\tau(t_0) + \mathbb{Z})$ for any fixed $t_0 \in \Delta$. Assume S is given by the holomorphic map $z(t)$; since S is a torsion section, one of its multiples must be 0, i.e. must be contained inside $\mathbb{Z}\tau(t) + \mathbb{Z}$. Therefore $z(t) \in \mathbb{Q}\tau(t) + \mathbb{Q}$, for all $t \in \Delta$. But S meets S_0 at some point, i.e. $z(t_0) = 0$ for some $t_0 \in \Delta$; since the $\mathbb{Q}\tau(t) + \mathbb{Q}$'s form a discrete subset of $\mathbb{C} \times \Delta$, $z(t)$ must be constantly 0, i.e. $S = S_0$.
Assume instead that S and S_0 intersect at a point of a semistable singular fiber I_n, with $n \geq 1$. As we did in Section 2.5, we can (locally) put the fibration in Jacobi form $\mathbb{C}^* \times \Delta/\{t^{nj} \mid j \in \mathbb{Z}\}$; in this description, the zero section S_0 has equation $z_0(t) \equiv 1$. Let S be given (locally) by the equation $z(t) : \Delta \to \mathbb{C}^*$; since S and S_0 meet, we can assume that $z(0) = 1$. S is a torsion section, thus there exists a $k \geq 1$ such that $z(t)^k = t^{nj}$ for some $j \in \mathbb{Z}$. Therefore $z(t)$ is locally a branch of $t^{\frac{nj}{k}}$, but since $z(t)$ is holomorphic, it must be the product of a root of unity with a non-negative integer power m of t. Using that $z(0) = 1$, necessarily $t = 0$ and $z(t)$ is identically 1, i.e. $S = S_0$ again. $\square$

We conclude with an alternative, more analytic way to study the local groups G_{R_c}. We follow the notations introduced by Kodaira in [Kod60].

Let $X^{\#}$ be the set of points of the elliptic surface X that are not critical for the fibration π: we are just deleting the vertical components of X with multiplicity greater than one, and all the singular points of the fibers. We denote by $X_c^{\#} = X_c \cap X^{\#}$ the set of non-critical points of π contained in the fiber X_c. Now let $X_0^{\#}$ be the subset of $X^{\#}$ obtained by deleting all the multiplicity one vertical components not meeting the zero section S_0; similarly, we denote by $X_{c0}^{\#}$ the intersection $X_c \cap X_0^{\#}$, i.e. the component of X_c meeting S_0 without the singular points of X_c contained in it.

Quite interestingly, $X_c^{\#}$ has a natural group structure. For, we can work locally around X_c, hence we only consider its germ of the fiber; let $\mathscr{S}$ be the set of all (local) sections of π in it. Each section in $\mathscr{S}$ identifies a point in $X_c^{\#}$ by restriction, and $\mathscr{S}$ is naturally an abelian group: we can sum two (local) sections by summing the corresponding points in each smooth fiber, and taking the closure in the surface. However, the restriction map $\mathscr{S} \to X_c^{\#}$ can be not injective; to solve this problem, we take the subgroup $\mathscr{S}_{00}$ of $\mathscr{S}$ formed by the (local) sections meeting X_c at the same point as S_0, and we consider the quotient $\mathscr{S}/\mathscr{S}_{00}$. It is immediate to notice that the restriction map gives a 1-1 correspondence between the abelian group $\mathscr{S}/\mathscr{S}_{00}$ and $X_c^{\#}$, hence induces a well defined structure of abelian group on $X_c^{\#}$. Since the intersection point of two components is always singular, $X_{c0}^{\#}$ is the connected component of the identity (or equivalently the set $\mathscr{S}_0$ of sections in $\mathscr{S}$ meeting the same component as S_0); therefore the quotient $X_c^{\#}/X_{c0}^{\#}$ can be identified with the quotient of $\mathscr{S}/\mathscr{S}_{00}$ by $\mathscr{S}_0/\mathscr{S}_{00}$, i.e. with the finite abelian group $\mathscr{S}/\mathscr{S}_0$ parametrizing the multiplicity one components of X_c.
We have a local version of the previous map γ defined as

$$\gamma : \mathscr{S} \longrightarrow G_{R_c} :$$

we just interpret $\mathscr{S}$ as the local MW(X). Then the local counterpart of Proposition 2.6.10 shows:

Proposition 2.6.15. *Let $c \in C$. Then $X_c^{\#}/X_{c0}^{\#}$ is isomorphic to G_{R_c}.*

Proof. The same argument as in the proof of Proposition 2.6.10 shows that $\mathscr{S}_0 = \mathrm{Ker}(\gamma)$. Since any component of $X_c^{\#}$ is intersected by some local section, we have that (interpreting $X_c^{\#}$ as $\mathscr{S}$) γ is surjective. $\square$

Consequently, we can update Table 2.10:

X_c	$X_{c0}^{\#}$	$G_{R_c} = X_c^{\#}/X_{c0}^{\#}$
I_0	I_0	$\{0\}$
I_n, $n \geq 1$	$\mathbb{C}^*$	$\mathbb{Z}/n\mathbb{Z}$
I_n^*, n even	$\mathbb{C}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
I_n^*, n odd	$\mathbb{C}$	$\mathbb{Z}/4\mathbb{Z}$
IV, IV^*	$\mathbb{C}$	$\mathbb{Z}/3\mathbb{Z}$
III, III^*	$\mathbb{C}$	$\mathbb{Z}/2\mathbb{Z}$
II, II^*	$\mathbb{C}$	$\{0\}$

Table 2.11: Local description of $X_c^{\#}$.

The second column is clear: if X_c is of type I_n for some $n \geq 1$, then $X_{c0}^{\#}$ is a rational curve without two points, while if X_c is of type I_n^*, II, III, IV, IV^*, III^* or II^*, $X_{c0}^{\#}$ is a rational curve without a single point. Actually these are not only equality as sets, but even isomorphisms of abelian groups: we transform X into a Weierstrass fibration, and we use the fact that the nonsingular part of the nodal elliptic curve (respectively, of the cuspidal elliptic curve) is isomorphic as an abelian group to $\mathbb{C}^*$ (respectively $\mathbb{C}$). For a proof of this fact, see [Sil09, Proposition III.2.5].

All this explicit description of the local groups $X_c^{\#}$ is particularly useful thanks to the following result:

Proposition 2.6.16. *Let $c \in C$. Then the restriction map*

$$\mathrm{TMW}(X) \longrightarrow \mathrm{Tors}(X_c^{\#})$$

is injective.

Proof. Clearly the map is well defined, since a torsion section must meet $X_c^{\#}$ in a torsion point. The result follows from the fact that two distinct torsion sections cannot intersect. $\square$

This has some immediate consequences: if there exists at least a $c \in C$ such that $X_c^{\#}$ is torsion-free, then there cannot exist torsion sections on X. For instance this is true when there are singular fibers of type II or II^*, as Table 2.11 shows.

Chapter 3

Configurations of Kodaira Fibers

In this last chapter we investigate the problem of determining the possible configurations of singular fibers on special classes of elliptic surfaces: we will restrict our attention to rational and K3 elliptic surfaces, since they are more or less the only cases with a reasonable number of possiblilities. The problem has been studied by Beauville [Bea82], Miranda and Persson [Mir90], [Per90], [MP86], [MP89], from very different point of views; our aim is to present most of the techniques and the approaches involved.

3.1 Extremal Rational Elliptic Surfaces

We begin our study by analyzing the easiest possible elliptic surfaces: the rational elliptic surfaces. As usual, we assume the existence of a distinguished section (i.e., the zero section S_0) on our rational elliptic surface $\pi\colon X \to \mathbb{P}^1$. Moreover, we know by Theorem 2.3.8 that X is the blow-up of $\mathbb{P}^2$ at 9 points.
Recall that the canonical bundle $K_X = -F$ equals minus a fiber, and the fundamental line bundle $\mathscr{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$ (see Example 2.2.12); consequently we have $\chi_{top}(X) = 12$ by Corollary 2.3.3. Following the notations in Section 2.6, we denote by $U = \langle S_0, F\rangle$ the rank 2 unimodular sublattice of $\mathrm{NS}(X)$, with signature $(1,1)$; since X is the blow-up of $\mathbb{P}^2$ at 9 points, we have that $\mathrm{NS}(X)$ is a unimodular lattice of signature $(1,9)$, hence $U^\perp$ is a rank 8, unimodular, negative definite lattice. Moreover $U^\perp$ is even: if $D \in U^\perp$, then by the genus formula we have that $D^2 + K_X D$ is even, but $K_X = -F \in U$ and therefore $K_X D = 0$. It follows from Theorem 1.4.2 that $U^\perp$ is isomorphic (as a lattice) to the lattice E_8.

Our final aim is to classify every possible configuration of the singular fibers on X, but we start by restricting ourselves to a very special case, the case of *extremal* rational elliptic surfaces. If not otherwise stated, all elliptic surfaces we consider have a section.

Definition 3.1.1. An elliptic surface $\pi\colon X \to C$ is said *extremal* if

$$\rho = h^{1,1} = 2 + \mathrm{rk}(R).$$

Recall that R is the sublattice of $U^\perp$ generated by the components of the fibers not meeting the zero section S_0. By Proposition 2.6.9 we have that $\mathrm{MW}(X) \cong U^\perp/R$, hence:

Proposition 3.1.2. *An elliptic surface X is extremal if it has maximal Picard number $\rho = h^{1,1}$ and the Mordell-Weil group of sections $\mathrm{MW}(X)$ is finite (or equivalently R generates $U^\perp$ over $\mathbb{Q}$).*

Proof. First, a remark: it is always true that $\rho \le h^{1,1}$ from the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^{1,1}(X)$$

coming by the exponential sequence for algebraic varieties. This motivates the use of the term *maximal* in the statement. Now the assertion follows immediately from the Shioda-Tate formula and the remark above. □

When we consider only rational surfaces, we can interpret the concept of *extremality* in a much more concrete way. Notice that the equality $\rho = h^{1,1}$, i.e. the maximality of the Picard number, is always verified for rational elliptic surfaces.

Proposition 3.1.3. *A rational elliptic surface $\pi\colon X \to \mathbb{P}^1$ is extremal if and only if one of the following equivalent facts holds:*

1. *The group $\mathrm{Aut}_{\mathbb{P}^1}(X)$ of automorphisms of X over $\mathbb{P}^1$, i.e. automorphisms $u\colon X \to X$ such that the diagram*
$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & \mathbb{P}^1 & \end{tikzcd}$$
commutes, is finite.

2. *The number of representations of X as a blow-up of $\mathbb{P}^2$ is finite.*

3. *The number of smooth rational curves with negative self-intersection is finite.*

4. *The number of reduced irreducible curves with negative self-intersection is finite.*

Proof. The equivalence of the last 3 assertions is quite easy: clearly 4. $\Rightarrow$ 3. $\Rightarrow$ 2.; moreover, since X is a blow-up of $\mathbb{P}^2$, we have the implication 2. $\Rightarrow$ 3. Finally, if C is a reduced irreducible curve with negative self-intersection, then by the genus formula and the equality $K_X = -F$ we get $-2 \leq C^2 + CK_X \leq -1$, forcing C to be a smooth rational curve with self-intersection -1 or -2 (since the genus $g(C)$ is a non-negative integer).
Now we prove that X is extremal if and only if the first assertion holds. For, let $u \in \mathrm{Aut}_{\mathbb{P}^1}(X)$, and notice that, since $K_X = -F$, u must preserve the fibration π, i.e. must send fibers into fibers. Therefore the image $u(S_0)$ of the zero section is again a section S, hence the automorphism $\tau_S^{-1} \circ u$ fixes S_0, where τ_S is the translation by S (seen inside $\mathrm{MW}(X)$). This gives a map from $\mathrm{Aut}_{\mathbb{P}^1}(X)$ to the finite group of automorphisms fixing S_0, which can be interpreted as the finite group of automorphisms of the generic fiber. Now, if X is extremal, i.e. $\mathrm{MW}(X)$ is finite, clearly $\mathrm{Aut}_{\mathbb{P}^1}(X)$ is finite, too. Conversely, if $\mathrm{Aut}_{\mathbb{P}^1}(X)$ is finite, then also the translations τ_S are finite in number, i.e. $\mathrm{MW}(X)$ is finite.
To conclude the proof, we prove for example that X is extremal if and only if the third assertion holds. By the genus formula, a smooth rational curve C with negative self-intersection must satisfy $-1 \geq C^2 = CF - 2 \geq -2$. If $C^2 = -2$, then $CF = 0$, hence C is a vertical (-2)-curve, and it must be one of the (finite) components of reducible fibers. If instead $C^2 = -1$, then $CF = 1$, hence C is a section. We have just proved that the number of sections and the number of smooth rational curves with negative self-intersection differ by a finite number, and this is sufficient to conclude. □

The extremality condition we have imposed for the moment is quite strict: for instance it forces some constraints on the possible configurations of singular fibers:

Proposition 3.1.4. *Let $\pi\colon X \to \mathbb{P}^1$ be an extremal rational elliptic surface. Then:*

1. $d(R) = \prod_{c\in\mathbb{P}^1} d(R_c) = |\,\mathrm{MW}(X)|^2$, *where $d(R_c) = d_c$ can be found in the a, b, δ table 2.4. In particular the discriminant $d(R)$ is a perfect square.*

2. $\sum_{c\in\mathbb{P}^1} (\chi_c - r_c) = 4$, *where χ_c and r_c are defined as in the a, b, δ table 2.4. In particular X has:*
    - 4 *semistable singular fibers, or*

- 2 *semistable singular fibers and* 1 *unstable singular fiber, or*
- 2 *unstable singular fibers.*

Proof. The first assertion is immediate: R orthogonally decomposes as the sum of the local R_c's (recall that these lattices are trivial if the fiber X_c is smooth), hence the first equality follows. For the second, by Remark 1.4.6 we have that $d(R) = d(U^\perp)[U^\perp : R]^2 = [U^\perp : R]^2$, thus we only have to recall that $\mathrm{MW}(X) \cong U^\perp/R$.
The second assertion is even easier: we have that $\chi_{top}(X) = \sum_{c\in\mathbb{P}^1} \chi_c = 12$ and $\mathrm{rk}(R) = \sum_{c\in\mathbb{P}^1} r_c = 8$, thus $\sum_{c\in\mathbb{P}^1} (\chi_c - r_c) = 4$, and we can just apply Proposition 2.2.21. □

These constraints let us write down all the (a priori) possible configurations of the fibers for the extremal rational elliptic surface X. If X has two unstable singular fibers, it is easy to see from that the a, b, δ table 2.4 that the only possibilities are

$$\{II, II^*\}, \quad \{III, III^*\}, \quad \{IV, IV^*\}, \quad \{I_0^*, I_0^*\}, \quad \{I_2^*, IV\}, \quad \{I_3^*, III\}, \quad \{I_4^*, II\},$$

since the sum of the two local χ_c's must be 12. Now, the two configurations $\{I_2^*, IV\}, \{I_3^*, III\}$ are impossible, since $d(R)$ is not a perfect square. The last configuration $\{I_4^*, II\}$ is also impossible, since it violates Theorem 2.2.26: the non-negative integer x would be equal to $-2 + \frac{1}{6}(10 - d)$, i.e. $d = \deg(j) \leq -2$, a contradiction.
Now assume that X has one unstable singular fiber and two semistable singular fibers; imposing the conditions $\sum_c \chi_c = 12$ and $\prod_c d_c$ equals a perfect square, we remain with the possibilities

$$\{II^*, I_1, I_1\}, \quad \{III^*, I_2, I_1\}, \quad \{IV^*, I_3, I_1\}, \quad \{IV, I_2, I_6\}, \quad \{III, I_1, I_8\}, \quad \{III, I_3, I_6\},$$

$$\{II, I_1, I_9\}, \quad \{II, I_2, I_8\}, \quad \{II, I_5, I_5\}, \quad \{I_4^*, I_1, I_1\}, \quad \{I_2^*, I_2, I_2\}, \quad \{I_1^*, I_4, I_1\}, \quad \{I_0^*, I_3, I_3\}.$$

Now we apply Theorem 2.2.26 to the possibilities number $4, 5, 6, 7, 8, 9, 13$, and we see that the non-negative integer x must be 0 in every case. Therefore the degree d of the j-map must be respectively $2, 3, 3, 4, 4, 4, 0$, and this contradicts Corollary 2.2.23.
Finally, if X has 4 semistable singular fibers, we remain with the possibilities $\{I_{n_1}, I_{n_2}, I_{n_3}, I_{n_4}\}$ such that $\sum_i n_i = 12$ and $\prod_i n_i$ equals a perfect square, i.e.

$$\{I_9, I_1, I_1, I_1\}, \quad \{I_8, I_2, I_1, I_1\}, \quad \{I_6, I_3, I_2, I_1\}, \quad \{I_5, I_5, I_1, I_1\}, \quad \{I_4, I_4, I_2, I_2\}, \quad \{I_3, I_3, I_3, I_3\}.$$

Applying Corollary 2.2.23 and Proposition 3.1.4 we obtain the following table:

Configuration	$\deg(j)$	$\lvert\mathrm{MW}(X)\rvert$	**Notation**
$\{II, II^*\}$	0	1	X_{22}
$\{III, III^*\}$	0	2	X_{33}
$\{IV, IV^*\}$	0	3	X_{44}
$\{I_0^*, I_0^*\}$	0	4	$X_{11}(j)$
$\{II^*, I_1, I_1\}$	2	1	X_{211}
$\{III^*, I_2, I_1\}$	3	2	X_{321}
$\{IV^*, I_3, I_1\}$	4	3	X_{431}

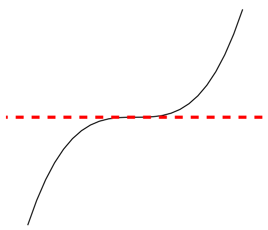
Configuration	$\deg(j)$	$\lvert \mathrm{MW}(X)\rvert$	**Notation**
$\{I_4^*, I_1, I_1\}$	6	2	X_{411}
$\{I_2^*, I_2, I_2\}$	6	4	X_{222}
$\{I_1^*, I_4, I_1\}$	6	2	X_{141}
$\{I_9, I_1, I_1, I_1\}$	12	3	X_{9111}
$\{I_8, I_2, I_1, I_1\}$	12	4	X_{8211}
$\{I_6, I_3, I_2, I_1\}$	12	6	X_{6321}
$\{I_5, I_5, I_1, I_1\}$	12	5	X_{5511}
$\{I_4, I_4, I_2, I_2\}$	12	8	X_{4422}
$\{I_3, I_3, I_3, I_3\}$	12	9	X_{3333}

Table 3.1: (A priori) possible configurations of the fibers for an extremal rational elliptic surface.

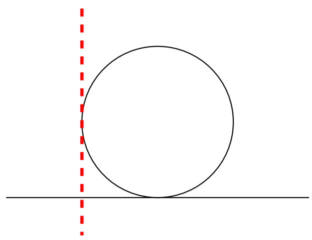
We haven't proven that these configurations can occur yet, and this will be the aim of the next part of the section. Our strategy will be completely concrete: we will exhibit pencils of plane cubics generating all the possible surfaces; for a different approach see [MP86]. Notice that we have already encountered some of these surfaces in the examples at the begininning of our exposition: for instance, Examples 2.1.8 and 2.1.9 are elliptic surfaces of types X_{11} and X_{9111}.

Remark 3.1.5. Recall the discriminant we introduced in Lemma 2.1.6; as we saw in the proof, it coincides with the usual discriminant. Therefore, if we have a pencil generated by the two cubics C_1, C_2, we can understand the singular fibers of the induced elliptic surface looking at the discriminant D of the moving cubic $\lambda C_1 + \mu C_2$: the singular fibers correspond to the zeroes of D, and their order is precisely the δ in the a, b, δ table 2.4. In the following, we will always write the discriminant up to constants.

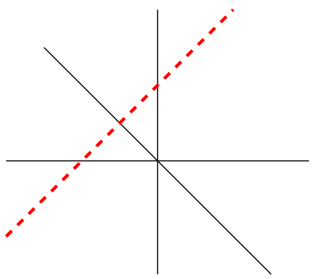
Let's start with the surfaces with 2 unstable singular fibers. We claim that they are generated by the pencils here below.



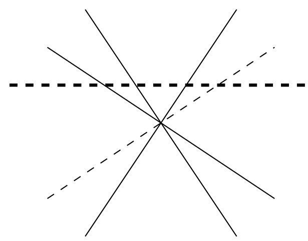
(a) $C_1 = \{yz^2 = x^3\}$, $C_2 = \{y^3 = 0\}$, $D = \lambda^{10}\mu^2$. This pencil generates an X_{22} surface.



(b) $C_1 = \{z(yz - x^2) = 0\}$, $C_2 = \{y^3 = 0\}$, $D = \lambda^9\mu^3$. This pencil generates an X_{33} surface.



(c) $C_1 = \{x(x-y)(x+y) = 0\}$, $C_2 = \{z^3 = 0\}$, $D = \lambda^8\mu^4$. This pencil generates an X_{44} surface.

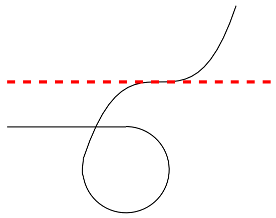


(d) $C_1 = \{x(x^2 - \alpha xz + z^2) = 0\}$, $C_2 = \{y^2 z = 0\}$, $D = \lambda^6\mu^6$. This pencil generates an X_{11} surface.

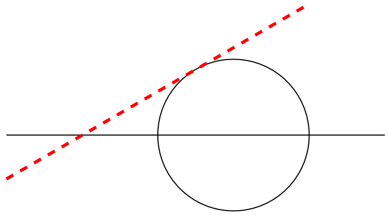
Figure 3.1: C_1 is the solid curve. D is the discriminant of the moving cubic $\lambda C_1 + \mu C_2$.

Looking at the discriminant we see that they all have exactly 2 singular fibers. For the first three ones, see Example 2.1.13: this shows that the fiber over C_2 is respectively a II^*, III^* and IV^* fiber. To understand the fiber over C_1 in the 3 cases, just notice that the base points are smooth points of C_1; therefore their strict transform is isomorphic to C_1 itself, hence it is respectively a II, III and IV fiber. The fourth pencil is resolved as in Example 2.1.8; notice that, varying appropriately $\alpha \neq \pm 2$, we obtain as many X_{11} surfaces as $\mathbb{C}$, one per each value of the j-invariant. These surfaces are not isomorphic, since their smooth minimal models (which are product surfaces) are not isomorphic. We will denote by $X_{11}(j)$ the X_{11} surface whose general fiber has j as j-invariant (see Table 3.1).

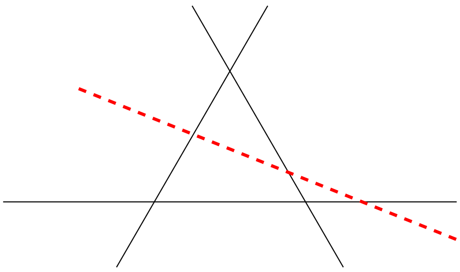
Now we pass to the elliptic surfaces with one unstable singular fiber and two semistable singular fibers. The pencils are given here below.



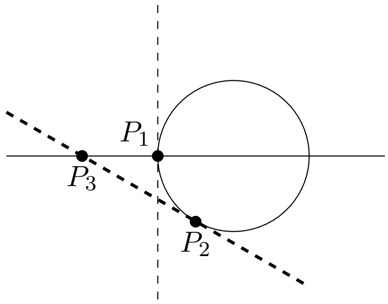
(a) $C_1 = \{yz^2 = x^3 - x^2y\}$, $C_2 = \{y^3 = 0\}$, $D = \lambda^{10}\mu(4\lambda + 27\mu)$. This pencil generates an X_{211} surface.



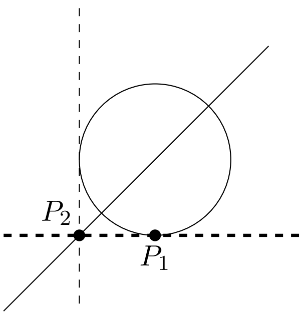
(b) $C_1 = \{x(x^2 + y^2 - z^2) = 0\}$, $C_2 = \{(x - z)^3 = 0\}$, $D = \lambda^9\mu^2(\lambda - 8\mu)$. This pencil generates an X_{321} surface.



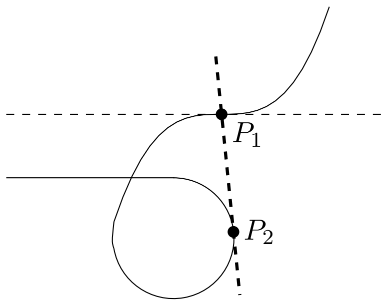
(c) $C_1 = \{(x + y)(x + z)(y + z) = 0\}$, $C_2 = \{x^3 = 0\}$, $D = \lambda^8\mu^3(8\lambda - 27\mu)$. This pencil generates an X_{431} surface.



(d) $C_1 = \{(y - x)(yz - x^2) = 0\}$, $C_2 = \{yz^2 = 0\}$, $D = \lambda^7\mu^4(\lambda - 16\mu)$. This pencil generates an X_{141} surface.



(e) $C_1 = \{(y - z)(yz - x^2) = 0\}$, $C_2 = \{yz^2 = 0\}$, $D = \lambda^8\mu^2(\lambda - \mu)^2$. This pencil generates an X_{222} surface.



(f) $C_1 = \{yz^2 = x^3 - x^2y\}$, $C_2 = \{y(x - y)^2 = 0\}$, $D = \lambda^{10}\mu(\lambda - 4\mu)$. This pencil generates an X_{411} surface.

Figure 3.2: C_1 is the solid curve. D is the discriminant of the moving cubic $\lambda C_1 + \mu C_2$.

Let's analyze one by one. The first one is similar to the one in Figure 3.1a; the only difference is

that C_1 is the nodal cubic curve, instead of the cuspidal one. Clearly this does not affect the fiber over C_2, which remains a II^* fiber. Since the other two singular fibers have $\delta = 1$, we see in the a, b, δ table 2.4 that they must be I_1 fibers.
The second one differs from the one in Figure 3.1b for the positioning of the two components of C_1: now the line intersects the conic in two distinct points, hence the fiber over C_1 is an I_2 fiber. As in Figure 3.1b we have that the fiber over C_2 is a III^* fiber, while the last one is an I_1 fiber as above.
For the third one we reason analogously: comparing it to the one in Figure 3.1c we see that the fiber over C_2 is a IV^* fiber, and it is clear that the fiber over C_1 is an I_3 fiber.

Now let us discuss the last 3 ones, that are a bit more delicate. In Figure 3.2d we have $P_1 = [0, 0, 1]$, $P_2 = [0, 1, 0]$, $P_3 = [1, 1, 0]$. First we resolve the pencil around P_1:

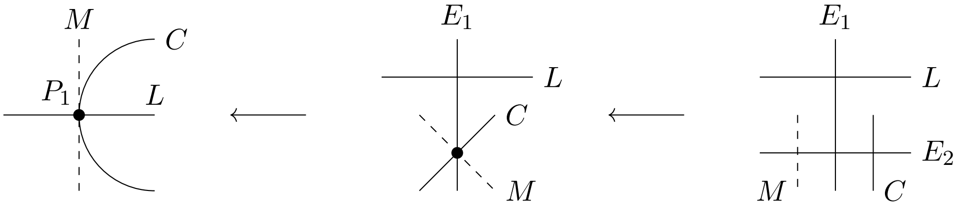


Figure 3.3: The marked point indicates where we are blowing up; the E_i are the corresponding exceptional divisors.

Therefore the fiber over C_1 is an I_4 fiber: the cycle of rational curves is given by L, E_1, E_2, C. Moreover blowing up the pencil around P_2 we obtain:

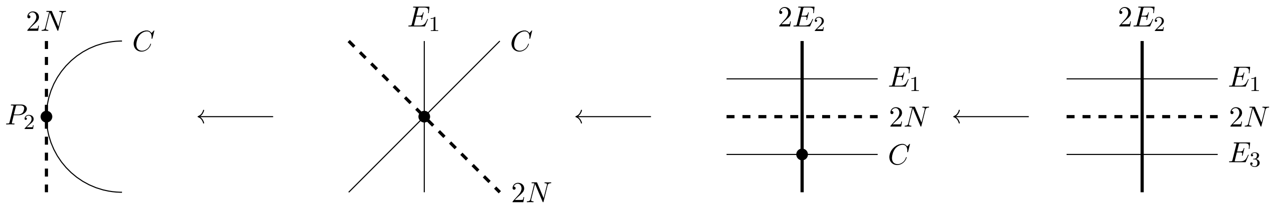


Figure 3.4: The marked point indicates where we are blowing up; the E_i are the corresponding exceptional divisors.

Since $2N$ intersects the other component of C_2 (M in Figure 3.3) and the exceptional divisor over P_3, we have that the fiber over C_2 is an I_1^* fiber.

Now switch to Figure 3.2e. Here $P_1 = [0, 1, 0]$ and $P_2 = [1, 0, 0]$. Clearly the fiber over C_1 is an I_2 fiber. The fiber over $[\lambda, \mu] = [1, 1]$ is I_2 too, since it has equation

$$y^2 z - x^2 y + x^2 z = 0 :$$

the only singular point is at $[0, 0, 1]$, and in the chart $z = 1$ it has a simple node at $(x, y) = (0, 0)$, because its quadratic part is $x^2 + y^2$. To understand the fiber over C_2, we have to blow up the pencil around P_1 and P_2. Around P_1 the pencil is analogous to Figure 3.4. Moreover around P_2 we have:

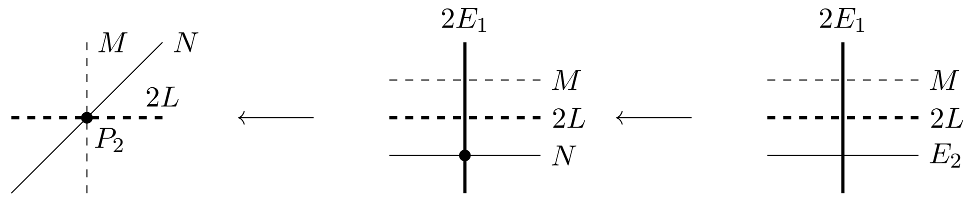


Figure 3.5: The marked point indicates where we are blowing up; the E_i are the corresponding exceptional divisors.

In conclusion, the fiber over C_2 contains three double lines (L, E_1 and one exceptional divisor over P_1), and 4 reduced lines (M, E_2 and two exceptional divisors over P_1), forming an I_2^* fiber.

Finally, consider Figure 3.2f. $P_1 = [0, 0, 1]$ is a flex point for C_1, and the double line $\{x = y\}$ is tangent to C_1 at $P_2 = [1, 1, 0]$. We only have to worry about the fiber over C_2; moreover we already know how to resolve the pencil around P_2, so let us focus on a neighbourhood of P_1. The resolution goes as below:

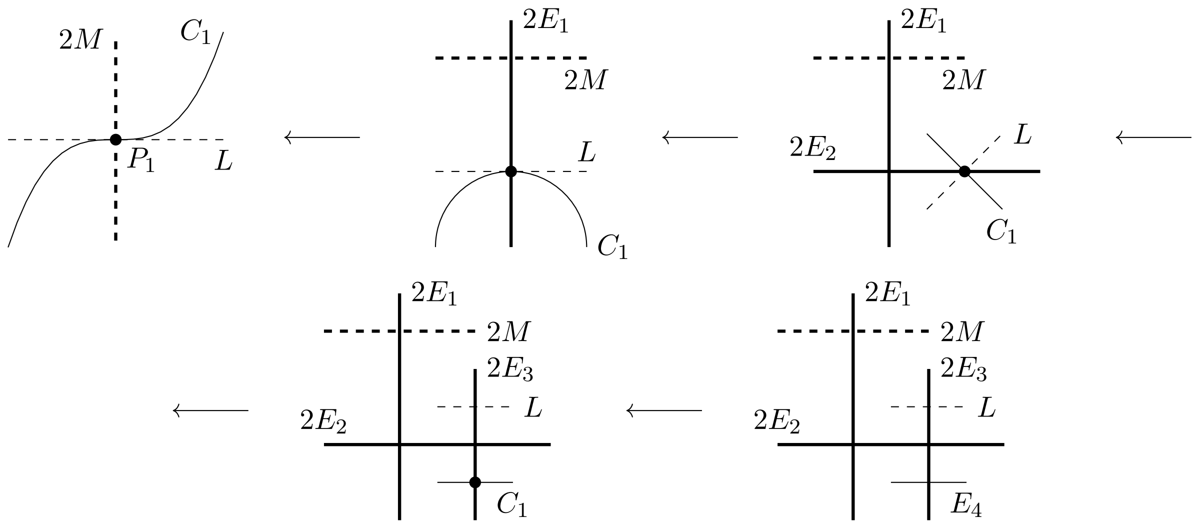


Figure 3.6: The marked point indicates where we are blowing up; the E_i are the corresponding exceptional divisors.

Therefore it is easy to see that the fiber over C_2 is an I_4^* fiber.

Now it only remains to prove the existence of the listed *semistable* surfaces, i.e. those surfaces with only semistable singular fibers. These 6 surfaces are sometimes called *Beauville surfaces*, as he was the first to classify and study them in [Bea82]; he used a more combinatorial approach though, while we prefer to provide explicit constructions.

First of all, we want to prove that the surface constructed in Example 2.1.9 is actually of type X_{9111}. We have already seen that a singular fiber is of type I_9, and the discriminant is

$$D = \lambda^9(27\lambda^3 + \mu^3);$$

therefore we have 3 simple singular fibers, which must be of type I_1.

Now we focus on X_{3333}. Consider the plane cubics C_1, C_2 given respectively by the equations $x^3 + y^3 + z^3 = 0$ and $xyz = 0$, and take the pencil generated by C_1 and C_2. In this case the figure cannot be too helpful, but the configuration is quite clear: C_1 intersects each edge of the triangle C_2 in three distinct points, far from the vertices. The discriminant is

$$D = \lambda^3(27\lambda^3 + \mu^3)^3,$$

hence we have 4 singular fibers with $\delta = 3$. The fiber over the triangle C_2 is clearly an I_3 fiber, since we have to blow up smooth points of C_2. The other 3 singular fibers come from the blow-up of the three isomorphic cubics given by the equations

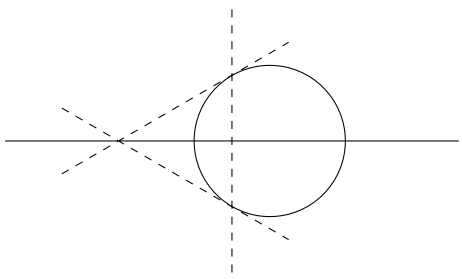
$$x^3 + y^3 + z^3 - 3\zeta_3^\alpha xyz = 0,$$

where $\alpha = 0, 1, 2$. It is immediate to see that they are isomorphic, since we pass from one to the other with the change of variables $(x', y', z') = (x\zeta_3^\beta, y, z)$ for an appropriate $\beta = 0, 1, 2$. Thus we want to study the plane cubic $\{x^3 + y^3 + z^3 - 3xyz = 0\}$. Its singular points are

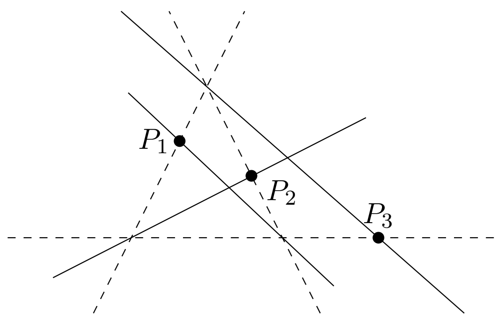
$$[1, 1, 1], \qquad [\zeta_3, \zeta_3^2, 1], \qquad [\zeta_3^2, \zeta_3, 1].$$

Since they all are away from the base points of the pencil, the strict transform of this cubic is isomorphic to the cubic itself, and since from the a, b, δ table 2.4 it can only be of type I_3 or III, the number of singular points assures us that the type of singular fiber is I_3.

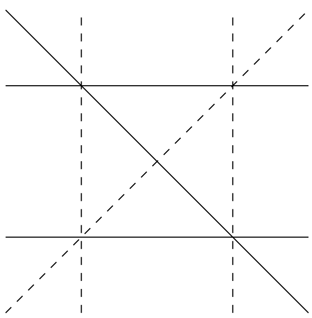
We claim that the last 4 surfaces are generated by the following pencils:



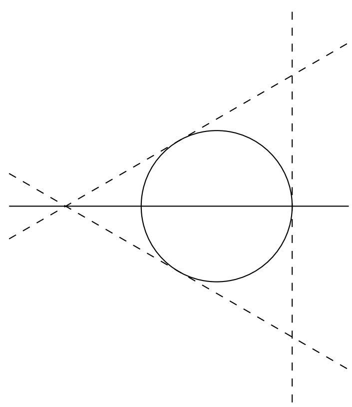
(a) $C_1 = \{(y - z)(yz - x^2) = 0\}$, $C_2 = \{xyz = 0\}$, $D = \lambda^8\mu^2(4\lambda - \mu)(4\lambda + \mu)$. This pencil generates an X_{8211} surface.



(b) $C_1 = \{(x+y)(y+z)(z+x) = 0\}$, $C_2 = \{xyz = 0\}$, $D = \lambda^6\mu^3(\lambda - \mu)^2(8\lambda + \mu)$. This pencil generates an X_{6321} surface. P_1, P_2, P_3 are collinear.



(c) $C_1 = \{x(x - z)(y - z) = 0\}$, $C_2 = \{yz(x - y) = 0\}$, $D = \lambda^5\mu^5(\lambda^2 + 11\lambda\mu - \mu^2)$. This pencil generates an X_{5511} surface.



(d) $C_1 = \{x(x^2 + 2yz + z^2) = 0\}$, $C_2 = \{z(x + y)(x - y) = 0\}$, $D = \lambda^4\mu^4(\lambda + \mu)^2(\lambda - \mu)^2$. This pencil generates an X_{4422} surface.

Figure 3.7: C_1 is the solid curve. D is the discriminant of the moving cubic $\lambda C_1 + \mu C_2$.

The first one is clear: the fiber over C_1 is an I_2 fiber, while the triangle C_2 is expanded into an octagon, as Example 2.1.9 shows (C_1 is tangent to the triangle at two vertices and passes simply through the third one).
We deal similarly with the second pencil: the fiber over C_1 is clearly of type I_3, and as above the one over C_2 is of type I_6. We have only to decide the type of singular fiber over $[1, 1]$; from the a, b, δ table 2.4 it can only be an I_2 or a II fiber. But the curve over $[1, 1]$ is given by the equation

$$(x + y)(y + z)(z + x) + xyz = 0, \text{ that is } \qquad (x + y + z)(xy + yz + xz) = 0,$$

hence it is reducible, and must be of type I_2. Actually, we could have proved this in a more explicit way: the two components $\{x + y + z = 0\}$ and $\{xy + yz + xz = 0\}$ are two smooth rational plane curves, and they intersect at the two points

$$P_1 = [1, \zeta_3, \zeta_3^2], \qquad P_2 = [1, \zeta_3^2, \zeta_3]$$

far from the base points of the pencil, where $\zeta_3 = \exp(\frac{2}{3}\pi i)$ is the primitive third root of unity.

The third pencil is easy: resolving the 4 vertices of the square we see that two of the four exceptional divisors belong to the fiber over C_1, while the other two belong to the fiber over C_2 (just notice that the exceptional divisor over a vertex belongs to the fiber having a double point at that vertex).
For the last pencil, notice that the triangle C_2 is expanded into a square, and the fiber over C_1 is of type I_4 (see Figure 3.3). Finally, the last two singular fibers (with $\delta = 2$) are given by the equations

$$x^3 + 2xy + x \pm x^2 = \pm y^2,$$

and after the change of variables $y' = y \mp x$ we obtain the equations

$$\pm(y')^2 = x^3 \pm 2x^2 + x = x(x \pm 1)^2,$$

and they both correspond to a nodal rational curve, hence they must contribute to fibers of type I_2.

We sum all this up in the next theorem:

Theorem 3.1.6. *The configurations listed in Table 3.1 exist, and they are realized by the following pencils of plane cubics:*

Surface	C_1	C_2
X_{22}	$yz^2 = x^3$	$y^3 = 0$
X_{33}	$z(yz - x^2) = 0$	$y^3 = 0$
X_{44}	$x(x - y)(x + y) = 0$	$z^3 = 0$
X_{11}	$x(x^2 - \alpha xz + z^2) = 0$	$y^2z = 0$
X_{211}	$yz^2 = x^3 - x^2y$	$y^3 = 0$
X_{321}	$x(x^2 + y^2 - z^2) = 0$	$(x - z)^3 = 0$
X_{431}	$(x + y)(x + z)(y + z) = 0$	$x^3 = 0$
X_{141}	$(y - x)(yz - x^2) = 0$	$yz^2 = 0$
X_{222}	$(y - z)(yz - x^2) = 0$	$yz^2 = 0$
X_{411}	$yz^2 = x^3 - x^2y$	$y(x - y)^2 = 0$

Surface	C_1	C_2
X_{9111}	$x^2y + y^2z + z^2x = 0$	$xyz = 0$
X_{8211}	$(y - z)(yz - x^2) = 0$	$xyz = 0$
X_{6321}	$(x + y)(y + z)(z + x) = 0$	$xyz = 0$
X_{5511}	$x(x - z)(y - z) = 0$	$yz(x - y) = 0$
X_{4422}	$x(x^2 + 2yz + z^2) = 0$	$z(x - y)(x + y) = 0$
X_{3333}	$x^3 + y^3 + z^3 = 0$	$xyz = 0$

Table 3.2: Extremal rational elliptic surface. The pencil is generated by C_1 and C_2.

With this work, we have proved that there exist exactly 16 configurations of singular fibers on extremal rational elliptic surfaces; moreover we have explicitly computed the degree of the j-map and the order of the (finite) group $\mathrm{MW}(X)$ in each case. Actually, it has been proved (see [MP86] and [Bea82]) that the presented surfaces are the unique models with those configurations of the singular fibers, except for the configuration $\{I_0^*, I_0^*\}$, which admits as many models as $\mathbb{C}$, one for each possible value of the j-map; we have listed these models in Table 3.2. This uniqueness argument is quite straightforward (but tedious) in the unstable cases, while it is more interesting and combinatorial in the semistable cases; in any case, we have decided to deal only with the existence of these configurations.

3.2 Rational Elliptic Surfaces

In the previous section we have analyzed the configurations of singular fibers for rational elliptic surfaces with a finite number of sections; in this one we want to drop the last condition and generalize the study to all rational elliptic surfaces. As you can understand, the amount of configurations we have to consider is far bigger, and the methods used before cannot help now. This means that we will use a more combinatorial approach, in order to deal with lots of configurations at once. It is worth remarking that this classification is also done in [Per90], with extremely concrete and geometric arguments; we prefer to follow the ideas of [Mir90], since we can obtain the results much faster. Clearly, in doing this we probably lose most of the beauty of the subject; however, we hope that the explicit and geometric approach used until now is sufficient to do justice to this fascinating world.

First, we want to recall some facts that will come in handy in the following pages. As usual, let $\pi\colon X \to \mathbb{P}^1$ be a rational elliptic surface with section. The topological Euler characteristic of X is 12, hence

$$\sum_c \chi_c = 12,$$

where χ_c is the topological Euler characteristic of the fiber X_c. Moreover, the lattice $R = \bigoplus_c^{\perp} R_c$ is a sublattice of $U^{\perp}$, and has rank ≤ 8; in other words, we have

$$r = \sum_c r_c \leq 8. \tag{1}$$

The case when equality holds is the *extremal* case we studied in the previous section: when this is the case, the product $D = \prod_c d_c$ must be a perfect square.

Recall that X can be seen as a double covering of a ruled surface R, i.e. we have a commutative diagram

$$\begin{tikzcd} & R \arrow[d, "q"] \\ X \arrow[ur, "g"] \arrow[r, "\pi"'] & \mathbb{P}^1 \end{tikzcd}$$

The double covering g is branched over a section of R and a trisection T. Since R is ruled over $\mathbb{P}^1$, we completely and explicitly know its Picard group; moreover R is constructed as the projectivization of the bundle $\pi_*\mathcal{O}_X(2S) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathscr{L}^{-2} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, hence R is isomorphic (over $\mathbb{P}^1$) to $\mathbb{F}_2$. The trisection T is a divisor with class inside $(q^*\mathscr{L}^6)(3) = \mathcal{O}_{\mathbb{P}^1}(6) \otimes \mathcal{O}_R(3)$, where $\mathcal{O}_R(1)$ is the tautological bundle of R; if we denote by h the class of $\mathcal{O}_R(1)$ inside $H^2(R,\mathbb{Z})$ and with f the class of a fiber, we have that (see for instance [Bea96, Chapter 3])

$$H^2(R,\mathbb{Z}) = \mathbb{Z}f \oplus \mathbb{Z}h, \quad fh = 1, \quad h^2 = -2, \quad [K_R] = -2h - 4f, \quad [T] = 3h + 6f.$$

The trisection T has class $6f + 3h$ inside $H^2(R,\mathbb{Z})$, and thus it has arithmetic genus

$$g(T) = 1 + \frac{T^2 + TK_R}{2} = 4.$$

Therefore, if X has at least a II, IV, IV^* or II^* fiber, from Table 2.4 we have that T intersects the corresponding fiber of R at a single point, which is unibranch for T. In particular T is irreducible, hence the genus drop γ due to the singularities on T must be ≤ 4 (since $g(T) - \gamma$ equals the geometric genus, which is non-negative). In fancy language, we can write this as

$$\text{if } (ii + iv + iv^* + ii^*) \geq 1, \quad \text{then} \qquad \gamma = \sum_c \gamma_c \leq 4, \tag{2}$$

where the γ_c's can be found in the a, b, δ table 2.4.

Now we turn to the j-map. The inequalities we are going to list are all proved at the end of Section 2.2. Recall that $j\colon \mathbb{P}^1 \to \mathbb{P}^1$ has degree

$$d = \deg(j) = \sum_{n\geq 1} n(i_n + i_n^*).$$

If the degree of j is 0, then j is constant and all singular fibers have the same j-invariant; in particular

$$\text{if } d = 0, \quad \text{then} \qquad \text{either } (ii + iv + iv^* + ii^*) = 0 \text{ or } (iii + iii^*) = 0. \tag{3}$$

The reason for the *either* is that, if X has $d = 0$ and no unstable singular fibers, it has no singular fibers at all. Remember also the inequalities

$$\text{if } d > 0, \quad \text{then} \qquad d - (ii + iv^*) - 2(iv + ii^*) \geq 0 \text{ and is multiple of } 3, \tag{4}$$

$$\text{if } d > 0, \quad \text{then} \qquad d - (iii + iii^*) \geq 0 \text{ and is multiple of } 2. \tag{5}$$

The last thing we have to recall about the ramification of the j-map is the so-called *extra* ramification of j: it is the non-negative integer

$$x = -2 + \frac{1}{6}\left[6\sum_{n\geq 1}(i_n + i_n^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - d\right],$$

which can be rewritten (using the explicit formula for the degree d above) as

$$x = \frac{1}{6}\left[\sum_{n\geq 1}(6 - n)(i_n + i_n^*) + 4(ii + iv^*) + 3(iii + iii^*) + 2(iv + ii^*) - 12\right] \in \mathbb{N}. \tag{6}$$

Now we are ready to consider our complete database of configurations of the fibers for X. For the moment we only impose the conditions about the topological Euler characteristic and the rank of the lattice R, that bound the number of configurations to 354: in particular we have $7, 50, 86, 81, 60, 34, 19, 10, 5, 2, 1$ possibilities respectively for configurations of length $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ (obviously the *length* of a configuration is the number of singular fibers in it). The configurations with 2 singular fibers have $\sum_c r_c = 8$, hence we have already studied them in the previous section. For the other cases, the remarks above give a quite long list of impossible configurations:

Configuration	**Reason for non-existence**	**Configuration**	**Reason for non-existence**
II, I_9, I_1	$(6):\ x=-1$	IV, II, I_6	$(6):\ x=-1$
I_3^*, I_2, I_1	$(1):\ r=8,\ D=8$	III, I_6, I_3	$(6):\ x=-1$
III, I_8, I_1	$(6):\ x=-1$	III, III, I_6	$(6):\ x=-1$
II, I_8, I_2	$(2):\ \gamma=5$	I_0^*, I_5, I_1	$(1):\ r=8,\ D=20$
II, II, I_8	$(6):\ x=-1$	I_0^*, I_4, I_2	$(1):\ r=8,\ D=32$
IV^*, I_2, I_2	$(1):\ r=8,\ D=12$	II, I_0^*, I_4	$(2):\ \gamma=5$
I_2^*, I_3, I_1	$(1):\ r=8,\ D=12$	IV, I_0^*, I_2	$(2):\ \gamma=5$
II, I_2^*, I_2	$(2):\ \gamma=5$	I_0^*, I_3, I_3	$(6):\ x=-1$
IV, I_7, I_1	$(1):\ r=8,\ D=21$	III, I_0^*, I_3	$(6):\ x=-1$
II, I_7, I_3	$(1):\ r=8,\ D=21$	II, I_5, I_5	$(6):\ x=-1$
III, I_7, I_2	$(1):\ r=8,\ D=28$	III, I_5, I_4	$(1):\ r=8,\ D=40$
III, II, I_7	$(6):\ x=-1$	IV, I_5, I_3	$(1):\ r=8,\ D=45$
I_1^*, I_3, I_2	$(1):\ r=8,\ D=24$	IV, III, I_5	$(6):\ x=-1$
II, I_6, I_4	$(1):\ r=8,\ D=24$	IV, I_4, I_4	$(1):\ r=8,\ D=48$
IV, I_6, I_2	$(6):\ x=-1$	IV, IV, I_4	$(6):\ x=-1$

Table 3.3: Impossible configurations with 3 singular fibers. The numbers in the parentheses in the last column indicate the remark we are using.

Configuration	**Reason for non-existence**	**Configuration**	**Reason for non-existence**
I_7, I_3, I_1, I_1	$(1):\ r=8,\ D=21$	I_5, I_3, I_2, I_2	$(1):\ r=8,\ D=60$
I_7, I_2, I_2, I_1	$(1):\ r=8,\ D=28$	I_4, I_4, I_3, I_1	$(1):\ r=8,\ D=48$
I_6, I_4, I_1, I_1	$(1):\ r=8,\ D=24$	II, I_4, I_4, I_2	$(2):\ \gamma=5$
I_6, I_2, I_2, I_2	$(1):\ r=8,\ D=48$	IV, I_4, I_2, I_2	$(2):\ \gamma=5$
II, I_6, I_2, I_2	$(2):\ \gamma=5$	I_4, I_3, I_3, I_2	$(1):\ r=8,\ D=72$
II, I_0^*, I_2, I_2	$(2):\ \gamma=5$	IV, IV, III, I_1	$(4):\ d=1$
I_5, I_4, I_2, I_1	$(1):\ r=8,\ D=40$	IV, IV, II, I_2	$(4):\ d=2$
I_5, I_3, I_3, I_1	$(1):\ r=8,\ D=45$	IV, III, III, II	$(3):\ d=0$

Table 3.4: Impossible configurations with 4 singular fibers. The numbers in the parentheses in the last column indicate the remark we are using.

Configuration	**Reason for non-existence**
II, I_4, I_2, I_2, I_2	$(2):\ \gamma=5$
IV, IV, II, I_1, I_1	$(4):\ d=2$
IV, III, II, II, I_1	$(4):\ d=1$
IV, I_2, I_2, I_2, I_2	$(2):\ \gamma=5$

Configuration	**Reason for non-existence**
IV, II, II, II, I_2	$(4):\ d=2$
III, III, III, II, I_1	$(5):\ d=1$
III, III, II, II, II	$(3):\ d=0$

Table 3.5: Impossible configurations with 5 singular fibers. The numbers in the parentheses in the last column indicate the remark we are using.

Configuration	Reason for non-existence
IV, II, II, II, I_1, I_1	$(4): \ d=2$
III, II, II, II, II, I_1	$(4): \ d=1$
$II, I_2, I_2, I_2, I_2, I_2$	$(2): \ \gamma=5$
II, II, II, II, II, I_2	$(4): \ d=2$
$II, II, II, II, II, I_1, I_1$	$(4): \ d=2$

Table 3.6: Impossible configurations with 6 or 7 singular fibers. The numbers in the parentheses in the last column indicate the remark we are using.

In total, we have discarted (counting also the work in the previous section) 61 configurations, and we remain with 293; it's our goal now to reject the configurations listed below, which pass the tests above:

Number	Configuration	Number	Configuration
1	II, I_6, I_3, I_1	8	IV, IV, I_3, I_1
2	I_0^*, I_3, I_2, I_1	9	IV, II, I_3, I_3
3	II, II, I_5, I_3	10	III, I_3, I_3, I_3
4	III, I_5, I_2, I_2	11	I_5, I_2, I_2, I_2, I_1
5	III, I_4, I_4, I_1	12	I_4, I_3, I_3, I_1, I_1
6	IV, I_4, I_3, I_1	13	II, I_3, I_3, I_3, I_1
7	II, I_4, I_3, I_3	14	I_3, I_3, I_2, I_2, I_2

Table 3.7: Configurations we want to reject.

We can immediately prove that the configuration number 2 is impossible: if it existed, then we could perform a quadratic twist to the I_0^* and I_3 fibers and obtain another elliptic surface with 3 singular fibers, of types I_3^*, I_2 and I_1. But this is impossible, as Table 3.3 shows.

In order to exclude some of the listed configurations, we want to introduce a general approach that uses the theory of lattices. We have collected the results we'll need in Section 1.4, hence we are only going to recall the most important ones.

In the previous section we have proved that the lattice $U^\perp$ is a unimodular even lattice, isomorphic to E_8; therefore, our lattice R must embed into E_8. There is a quite classical way to decide if the lattice R can embed into the E_8 lattice, that uses the discriminant form group:

Proposition 3.2.1. *Let $R = \bigoplus_c R_c \subseteq U^\perp$ be the lattice associated to the singular fibers of the rational elliptic surface X. Then there exists a q_R-isotropic subgroup $H < G_R$ such that the length of the abelian group $H^\perp/H$ is at most $8-r$, where $r = \mathrm{rk}(R)$.*

Proof. Recall that G_R is the quotient $R^\#/R$, and q_R is the quadratic form on G_R such that $q_R(x) = \frac{1}{2}\langle x, x\rangle \pmod{\mathbb{Z}}$. By Corollary 1.4.8 we have that $G_{R^\perp} \cong G_{R^{\perp\perp}}$; moreover $\mathrm{rk}(R) = \mathrm{rk}(R^{\perp\perp})$, thus by Proposition 1.4.5 there exists a q_R-isotropic subgroup $H < G_R$ such that $G_{R^\perp} \cong G_{R^{\perp\perp}} \cong H^\perp/H$. Since $\mathrm{rk}(R^\perp) = 8-r$, we have that the length of the abelian group $G_{R^\perp} \cong H^\perp/H$ must be $\leq 8-r$.- $\square$

In this setup, Proposition 1.4.3 acquires great importance:

Corollary 3.2.2. *The lattices*

$$A_2 \oplus A_2 \oplus A_3, \qquad A_1 \oplus A_1 \oplus A_1 \oplus A_4, \qquad A_1 \oplus A_1 \oplus A_1 \oplus A_2 \oplus A_2$$

cannot embed into E_8.

Proof. We use the previous proposition and we show that these 3 lattices have no non-zero isotropic elements inside the discriminant form group. This is quite easy: by Remark 1.4.4, G_R has non-trivial isotropic elements only if its Sylow subgroups have some, and we use Proposition 1.4.3. □

This allows us to rule out the configurations number $4, 6, 7, 11, 12, 14$ (recall Table 2.10). Unfortunately, our work is not complete yet, as we have to remove other 7 configurations. Thus we are going to introduce our last tool, that again uses the extreme rigidity of the j-map.

Let C be any smooth projective curve, and let $f: C \to \mathbb{P}^1$ be a branched covering of degree d. Let $Q_1, \ldots, Q_k$ be the branch points, and denote $\widetilde{\mathbb{P}}^1 = \mathbb{P}^1 \backslash \{Q_1, \ldots, Q_k\}$. An easy topological argument shows that, if Q is any point in $\widetilde{\mathbb{P}}^1$, there exist loops α_i based at Q, homotopic to small loops around Q_i, such that they generate $\pi_1(\widetilde{\mathbb{P}}^1, Q)$ and such that they satisfy the only relation $\alpha_1 \cdot \ldots \cdot \alpha_k = 1$. Then we can label the d points in $f^{-1}(Q)$ (the labeling is irrelevant), and consider the *monodromy action* of $\pi_1(\widetilde{\mathbb{P}}^1, Q)$ on $f^{-1}(Q)$: if α is any loop in $\pi_1(\widetilde{\mathbb{P}}^1, Q)$, then we can lift it to a path inside $\widetilde{C} = f^{-1}(\widetilde{\mathbb{P}}^1)$, starting at some fixed $P \in f^{-1}(Q)$ and ending at some other point $\mu_\alpha(P) \in f^{-1}(Q)$. This defines a homomorphism

$$\begin{array}{cccc} \mu: & \pi_1(\widetilde{\mathbb{P}}^1, Q) & \longrightarrow & S_d = S(f^{-1}(Q)) \\ & \alpha & \longmapsto & \mu_\alpha \end{array}$$

and this is clearly independent from the choice of the base point Q. Our generators α_i are mapped into permutations $\sigma_i = \mu_{\alpha_i} \in S_d$ such that $\sigma_1 \cdot \ldots \cdot \sigma_k = 1$. Actually, we can say a lot more about these permutations (see [Mir95, Chapter III.4]):

Proposition 3.2.3. *Let $\sigma_1, \ldots, \sigma_k \in S_d$ be constructed as above. Then they generate a transitive subgroup of S_d, and the cycle structure of σ_i is as follows: it contains $|f^{-1}(Q_i)|$ cycles, one for each $P \in f^{-1}(Q_i)$, and the length of the cycle corresponding to P is exactly the ramification index e_P.*

We will say that a set $\sigma_1, \ldots, \sigma_k \in S_d$ of permutations satisfying $\sigma_1 \cdot \ldots \cdot \sigma_k = 1$ and the properties of the proposition is a *Hurwitz factorization* of type $(d, T_1, \ldots, T_k)$, where T_i is the cycle structure of σ_i. What is really outstanding is that these factorizations classify all possible branched covers $f: C \to \mathbb{P}^1$ (see again [Mir95]):

Theorem 3.2.4. *The monodromy induces a bijective map between Hurwitz factorizations of type $(d, T_1, \ldots, T_k)$ and branched coverings $f: C \to \mathbb{P}^1$ of degree d, with branch points $Q_1, \ldots, Q_k$, such that the partition $\{e_P \mid P \in f^{-1}(Q_i)\}$ of d is exactly T_i.*
Moreover, a relabeling of the preimages of Q corresponds to composing the branched covering f with a biholomorphism of C.

Now we can return to our configurations and lose generality again: C is simply $\mathbb{P}^1$, and in 6 of the 7 remaining configurations (precisely, all except the number 13) we have $x = 0$, i.e. all the ramification of the j-map occurs above $0, 1, \infty$. Therefore, proving the impossibility of these configurations is equivalent to proving the non-existence of certain permutations, as stated above. We will indicate with $\sigma_0, \sigma_1, \sigma_\infty$ respectively the monodromy permutations over $0, 1, \infty$.

In the configuration number 1 we would have $d = 10$: clearly σ_∞ has cycle structure 136 (this means that σ_∞ contains a 1-cycle, a 3-cycle and a 6-cycle). Moreover the only singular fiber with $j = 0$ is the unique II fiber, which must have multiplicity $m = 1$ since there is no extra ramification (recall Table 2.5). Therefore the other ramification at $j = 0$ must occur in correspondence to smooth fibers, that again by Table 2.5 must have multiplicity $m = 3$; hence the cycle structure of σ_0 has to be 13^3 (i.e. 1333 in the previous notation). Finally, a similar argument shows that σ_1 has cycle structure 2^5.
We can deal similarly with the other 5 configurations to decide the cycle structure of the 3 permutations in each case: they are collected together here below.

Number	**Configuration**	d	σ_0	σ_1	σ_∞
1	II, I_6, I_3, I_1	10	13^3	2^5	136
3	II, II, I_5, I_3	8	$1^2 3^2$	2^4	35
5	III, I_4, I_4, I_1	9	3^3	12^4	14^2
8	IV, IV, I_3, I_1	4	2^2	2^2	13
9	IV, II, I_3, I_3	6	123	2^3	3^2
10	III, I_3, I_3, I_3	9	3^3	12^4	3^3

Table 3.8: Cycle structure for the monodromy permutations of 6 impossible configurations.

Finally, proving the impossibility of these 6 configurations can be deduced from the following:

Proposition 3.2.5. *There are not Hurwitz factorizations formed by* 3 *permutations with cycle structures as in Table 3.8.*

Proof. In each case, put $\alpha = \sigma_0$, $\beta = \sigma_1$ and $\gamma = \sigma_\infty^{-1}$. We are going to prove that there are not permutations $\alpha, \beta, \gamma \in S_d$ with those cycle structures, with $\gamma = \alpha\beta$, generating a transitive subgroup of S_d. Keep in mind the following remark: if a 2-cycle of β is contained in a 3-cycle of α, then the product must have a fixed point (for instance, $(abc)(ab)$ fixes b).

8. This is easy: α and β are $2+2$-cycles, thus also γ is a $2+2$-cycle, a contradiction.

9. Assume $\alpha = (1)(23)(456)$. Since γ has no fixed points (it is a $3+3$ cycle), by the remark none of the 2-cycles of β are contained in $\{4,5,6\}$. Therefore (12) or (13) cannot be 2-cycles of β, hence we assume that a 2-cycle of β is (14). Again, we may assume that $\beta = (14)(25)(36)$; but then $\gamma = (1534)(26)$ has the wrong cycle structure.

3. Assume $\alpha = (1)(2)(345)(678)$. α and β generate a transitive subgroup of S_8, thus β cannot contain (12). Without loss of generality, β contains (13). By the remark, β cannot contain (24) or (25) (otherwise γ would have a fixed point), thus we assume that β contains (26). Again, by the remark we have only two possibilities for β: (13)(26)(47)(58) or (13)(26)(48)(57). However, in the first case we would have $\gamma = (1483)(2756)$, while in the second $\gamma = (146273)(58)$, and they are both contradictions.

5. Assume $\alpha = (123)(456)(789)$, and that β fixes 1. β cannot contain (23), otherwise the subgroup generated by α, β would stabilize $\{1,2,3\}$; hence we assume that β contains (24). Now β cannot contain (35), otherwise $\gamma = \alpha\beta$ would contain the 3-cycle (125); moreover, if β contains (36), then γ contains the 2-cycle (34), a contradiction. Therefore we may assume that β contains (37). We remain with 3 cases for β:

$$(1)(24)(37)(56)(89), \qquad (1)(24)(37)(58)(69), \qquad (1)(24)(37)(59)(68).$$

   None of these possibilities work: in the first case γ has two fixed points (6 and 9), in the second γ is a 8-cycle, and in the third γ contains the 2-cycle (69).

10. Analogous to the previous one.

1. Assume $\alpha = (0)(123)(456)(789)$, and that β contains (01). By transitivity, β cannot contain (23), hence we assume that it contains (24). Moreover β cannot contain (35) or (36): in the first case, γ would contain the 4-cycle (1025), while in the second γ would contain the 2-cycle (34). Thus β contains (37). Now $\gamma = \alpha\beta$ acts repeatedly on 7 and 4 as

$$7 \to 1 \to 0 \to 2 \to 5, \qquad 4 \to 3 \to 8,$$

and since γ must contain a 3-cycle, necessarily this 3-cycle is (438), i.e. β sends 8 to 6. Consequently $\beta = (01)(24)(37)(59)(68)$, hence γ contains the 2-cycle (69), a contradiction.

□

The proof is not so illuminating, but this technique involving monodromy permutations can be used to refute configurations of singular fibers in an algorithmic way. Unfortunately, for the moment this approach can only help us to prove the impossibility of configurations, not the existence; however, in the last part of the section we will see how to use this to construct elliptic surfaces with prescribed singular fibers. Before talking about existence issues, we have to conclude our work and prove that the last configuration $\{II, I_3, I_3, I_3, I_1\}$, with number 13 in Table 3.7, does not exist.

The problem with this configuration is that it has $x = 1$, i.e. it has an *extra* ramification point which is not accounted for by the singular fibers. This point can be over $0, 1$ or another $P \neq 0, 1, \infty$. If it is over 0, it can be the point corresponding to the II fiber (that would have multiplicity $m = 4$, see Table 2.5) or to a smooth fiber (that would have multiplicity $m = 6$); if it is over 1, it must be a point corresponding to a smooth fiber (that has multiplicity 4); if it is over $P \neq 0, 1, \infty$, it must be again a point corresponding to a smooth fiber (that has multiplicity 2). Therefore we have 4 possibilities for the cycle structure of the monodromy permutations:

- $\sigma_0 = 3^2 4$, $\sigma_1 = 2^5$, $\sigma_\infty = 13^3$;
- $\sigma_0 = 136$, $\sigma_1 = 2^5$, $\sigma_\infty = 13^3$;
- $\sigma_0 = 13^3$, $\sigma_1 = 2^3 4$, $\sigma_\infty = 13^3$;
- $\sigma_0 = 13^3$, $\sigma_1 = 2^5$, $\sigma_\infty = 13^3$, $\sigma_P = (1^8 2)$.

Proposition 3.2.6. *There are not three permutations $\sigma_0, \sigma_1, \sigma_\infty \in S_{10}$ with cycle structure as in the first 3 cases above such that $\sigma_0 \cdot \sigma_1 \cdot \sigma_\infty = 1$. Moreover, the only 4 permutations $\sigma_0, \sigma_1, \sigma_\infty, \sigma_P \in S_{10}$ with cycle structure as in the last case above such that $\sigma_0 \cdot \sigma_P \cdot \sigma_1 \cdot \sigma_\infty = 1$ do not generate a transitive subgroup of S_{10}.*

We avoid the proof of this proposition, as it analogous to the previous one; actually, we could check this with the help of a computer, but it wouldn't add much to this thesis. A complete proof can be found in [Mir90].

This concludes the work to exclude the last impossible configuration, and we remain with a list of 279 configurations. Now it is the moment to prove that these ones actually exist.

We begin with a definition. Let $j\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a map with degree $d = \deg(j) > 0$; we say that j *belongs* to a given list of singular fibers if it has:

over 0: $(ii + iv^*)$ points of multiplicity 1, $(iv + ii^*)$ points of multiplicity 2, $\frac{1}{3}[d - (ii + iv^*) - 2(iv + ii^*)]$ points of multiplicity 3;

over 1: $(iii + iii^*)$ points of multiplicity 1, $\frac{1}{2}[d - (iii + iii^*)]$ points of multiplicity 2;

over ∞: $(i_n + i_n^*)$ points of multiplicity n, $n \geq 1$.

Notice that this is precisely the desired ramification (over $0, 1, \infty$) for the j-map with that configuration of singular fibers.
Recall the elliptic surface we constructed with the identity as j-map, given by

$$y^2 = x^3 - 3t(t - s)s^2 + 2t(t - s)^2 s^3.$$

It has 3 singular fibers: a II fiber over $[0,1]$, a III fiber over $[1,1]$ and an I_1^* fiber over $[1,0]$. If we perform a quadratic twist over $[1,1]$ and $[1,0]$, we obtain another elliptic surface with the identity as j-map, given by

$$y^2 = x^3 - 3t(t-s)^3 + 2t(t-s)^5$$

(we have multiplied A by $\frac{(t-s)^2}{s^2}$ and B by $\frac{(t-s)^3}{s^3}$), with a II fiber over $[0,1]$, a III^* fiber over $[1,1]$ and an I_1 fiber over $[1,0]$. Now, if we perform a base change of order m at 0, 1 or ∞, then the singular fibers change as in Table 2.8; in particular, we can obtain singular fibers of any type, up to a $*$; this motivates the following:

Theorem 3.2.7. *Suppose that a list of singular fibers is given, satisfying properties* $(1)-(6)$ *at the beginning of the section, with* $d = \deg(j) > 0$. *Suppose further that a map* $j\colon \mathbb{P}^1 \to \mathbb{P}^1$ *exists, belonging to the list of singular fibers. Then there exists a rational elliptic surface with the prescribed list of singular fibers.*

Proof. First, write

$$d = \sum_{n\geq 1} (i_n + i_n^*) = (ii+iv)^* + 2(iv+ii^*) + 3a = (iii+iii^*) + 2b.$$

Let Y be the pull-back via j of the elliptic surface with identity as j-map, given by

$$y^2 = x^3 - 3t(t-s)s^2 + 2t(t-s)^2s^3.$$

By Table 2.8, Y has the following singular fibers: over the points over $j=0$, it has $(ii+iv^*)$ II fibers, $(iv+ii^*)$ IV fibers, and a I_0^* fibers; over the points over $j=1$, it has $(iii+iii^*)$ III^* fibers, and b I_0^* fibers; over the points over $j=\infty$, it has $(i_n+i_n^*)$ I_n fibers for each $n\geq 1$.
In total, Y has $(a+b+iii+iii^*)$ $*$-fibers. Since we want our elliptic surface to be rational, necessarily there is at most one $*$-fiber in the prescribed list of singular fibers; let $e\in\{0,1\}$ be this number of $*$-fibers. Our goal is to "deflate" the exceeding $*$-fibers, but we have to be careful about the evenness of the number $(a+b+iii+iii^*)$. To do so, notice that from the equality $12 = \sum_c \chi_c$ we have

$$2(ii+iv^*) + 3(iii+iii^*) + 4(iv+ii^*) + 6e + d = 12,$$

thus

$$\begin{aligned} a+b+iii+iii^* &= \frac{1}{3}[(12-2(ii+iv^*)-3(iii+iii^*)-4(iv+ii^*)-6e)-(ii+iv^*)-2(iv+ii)^*]+\\ &\quad+\frac{1}{2}[(12-2(ii+iv^*)-3(iii+iii^*)-4(iv+ii^*)-6e)-(iii+iii^*)]+iii+iii^* =\\ &= 10-5e-2(ii+iv^*)-2(iii+iii^*)-4(iv+ii)^*, \end{aligned}$$

hence $(a+b+iii+iii^*)$ is even if and only if e is even. Therefore we can perform a quadratic twist to the $(a+b+iii+iii^*-e)$ points corresponding to the exceeding $*$-fibers. If $e=0$, by construction we have the desired elliptic surface; if $e=1$, the only problem can be that the $*$-fiber is not the prescribed one, but we fix this performing an appropriate quadratic twist (to the $*$-fiber and the fiber which should have the $*$). In conclusion, we have constructed an elliptic surface X with the prescribed list of singular fibers; we have only to check that this surface is rational. Since by property (1) we have $\chi_{top}(X) = 12$, then $\deg(\mathscr{L}) = 1$ by Corollary 2.3.3, hence $P_2(X) = 0$ by Corollary 2.3.6. Moreover $q(X) = 0$ by Proposition 2.3.1 (X is not a product: it has singular fibers!), and by Castelnuovo's theorem X is rational. □

This shows that, in order to construct a rational elliptic surface with prescribed configuration of singular fibers, it is sufficient to construct an appropriate j-map. Since j is nothing more than a branched covering $\mathbb{P}^1 \to \mathbb{P}^1$, we can use Theorem 3.2.4 to reduce the problem to finding appropriate monodromy permutations. If the prescribed list of singular fibers has $x=0$, we know that all the ramification occurs over $0,1,\infty$, thus we have to find only 3 permutations:

Theorem 3.2.8. *Suppose that a list of singular fibers is given, satisfying properties* $(1)-(6)$ *at the beginning of the section, with* $d = \deg(j) > 0$ *and* $x = 0$. *Then a rational elliptic surface with this configuration of singular fibers exists if and only if there are three permutations* $\sigma_0, \sigma_1, \sigma_\infty \in S_d$ *such that:*

- σ_0 *contains* $(ii + iv^*)$ 1*-cycles,* $(iv + ii^*)$ 2*-cycles and* $\frac{1}{3}[d - (ii + iv^*) - 2(iv + ii^*)]$ 3*-cycles;*
- σ_1 *contains* $(iii + iii^*)$ 1*-cycles and* $\frac{1}{2}[d - (iii + iii^*)]$ 2*-cycles;*
- σ_∞ *contains* $(i_n + i_n^*)$ n*-cycles for each* $n \geq 1$*;*
- $\sigma_0 \cdot \sigma_1 \cdot \sigma_\infty = 1$*;*
- $\sigma_0, \sigma_1, \sigma_\infty$ *generate a transitive subgroup of* S_d.

Therefore we have reduced our problem of proving the existence of rational elliptic surfaces with prescribed singular fibers to a combinatorial exercise, that we can attack algorithmically (at least for the configurations with $x = 0$). This is an incredibly powerful tool, that we will exploit soon. For instance, we could use this to prove the uniqueness of the extremal rational elliptic surfaces with $d > 0$ found in the previous section, by showing the uniqueness of the appropriate monodromy permutations. For the moment, we want to show how we can deal with configurations with $x \geq 1$: it is only a matter of "deforming" configurations with $x = 0$. Let us be more explicit.

Suppose that a list of singular fibers exists, with $d = \deg(j) > 0$. The j-map is surely ramified over $0, 1, \infty$, and possibly elsewhere; this corresponds to a set $\sigma_0, \sigma_1, \sigma_\infty, \sigma_{P_1}, \ldots, \sigma_{P_k}$ of permutations in S_d. Notice the equality

$$(1, 2, \ldots, m) = (1, 2, \ldots, k)(k + 1, k + 2, \ldots, m)(k, m)$$

for any $1 \leq k < m$. Now suppose that this m-cycle appears in σ_0, σ_1 or σ_∞; suppose for instance that it is in σ_0 (notice that up to a relabeling we can suppose that this cycle is the last one in σ_0). We can perform the following operation: we replace the m-cycle in σ_0 with the k-cycle and the $(m - k)$-cycle, and we introduce a new permutation $\sigma_P = (k, m)$ just after σ_0, with $P \notin \{0, 1, \infty, P_1, \ldots, P_k\}$. By construction we have that $\sigma_0, \sigma_P, \sigma_1, \sigma_\infty, \sigma_{P_1}, \ldots, \sigma_{P_k}$ have the identity as product, and they generate a transitive subgroup of S_d; hence they correspond to a well-defined j-map, that has the same configuration of singular fibers as before, except above 0 (since $P \neq 0, 1, \infty$ must correspond to a smooth fiber). Notice that this process increases x by 1. We can reason similarly with 1 and ∞, and we obtain the following deformations:

Fiber	**Deforms into**
I_n, $n \geq 2$	$I_k + I_{n-k}$ over ∞, $1 \leq k < n$
I_0^*	$II + IV$ if over 0
I_0^*	$III + III$ if over 1
IV	$II + II$ over 0

Table 3.9: Deformations of singular fibers over $0, 1, \infty$.

Notice that the other deformations

$$I_n \to I_k + I_{n-k}^*, \quad I_0^* \to II + II^*, \quad I_0^* \to IV + IV^*, \quad I_0^* \to III + III^*, \quad IV \text{ (or } II^*) \to II + IV^*$$

are not possible since the sum of the Euler characteristics must be preserved.

Finally, we have all the necessary ingredients to state the following:

Theorem 3.2.9. *The remaining* 279 *configurations of singular fibers exist.*

Proof. As we remarked, if the degree d of the associated j-map is strictly greater than 0, we can construct all configurations with $x = 0$ by finding appropriate permutations in S_d; moreover, all the other configurations (with $x \geq 1$) can actually be obtained by deforming the configurations with $x = 0$, as shown above. The list of all the permutations in each case (that we are not going to write down for obvious reasons) can be found in [Mir90]. We remain with the configurations with constant j (i.e. $d = 0$). Some of them are constructed in the previous section, and others can be obtained by twisting these ones; the last 6

$$\{IV, IV, IV\}, \quad \{IV, IV, II, II\}, \quad \{IV, II, II, II, II\}, \quad \{III, III, III, III\}, \quad \{II, II, II, II, II, II\}$$

can be easily constructed by writing down their Weierstrass equations. For instance

$$y^2 = x^3 + s^2t^2(s-t)^2$$

defines a rational elliptic surface with 3 singular IV fibers, while

$$y^2 = x^3 + st(s-t)(s-2t)(s-3t)(s-4t)$$

defines a rational elliptic surface with 6 singular II fibers; the others are obtained analogously. □

3.3 Semistable Elliptic K3 Surfaces

In this last section we want to investigate the configurations of singular fibers on elliptic K3 surfaces. Although the reader could point out that this is a pointless algorithmic exercise, it allows us to introduce more fascinating techniques and approaches that exploit the theory we have developed throughout the thesis. We immediately warn that a complete study as in the previous two sections is practically impossible, as the possible configurations exceed the tens of thousands. Therefore we limit ourselves to study the possible configurations of *semistable* singular fibers. There are at least two good reasons to consider these fibers: first of all they are the "general" type of singular fibers, and, secondly, all elliptic surfaces can be carried into a *semistable* elliptic surface after a finite number of appropriate base changes.

Let $\pi\colon X \to C$ be an elliptic K3 surface. By Corollary 2.3.7, we have that $C = \mathbb{P}^1$ and $\deg(\mathscr{L}) = 2$, hence $\chi_{top}(X) = 24$. Therefore the degree d of the j-map is at most 24, and our assumption to consider only *semistable* elliptic K3 surfaces is equivalent to impose that d is exactly 24. By the Shioda-Tate formula 2.6.8 we have

$$\rho = 26 + \mathrm{rk}(\mathrm{MW}(X)) - s,$$

where s is the number of singular fibers, and since $\rho \leq h^{1,1}(X) = 20$, we have that there must exist at least 6 singular fibers. If in particular $s = 6$, then the Mordell-Weil group is finite, and we will refer to this case as the *extremal* case (notice the analogy with *extremal* rational elliptic surfaces).

Our database is thus formed by all the s-tuples $\{n_1, \dots, n_s\}$ of positive integers with $s \geq 6$ and $\sum n_i = 24$; a simple computation shows that there are 1242 possible configurations, of which 199 with 6 singular fibers (the *extremal* cases). We beware the reader that we will not dwell much on the existence of the possible configurations, as it is a rather tedious combinatorial exercise; we will merely recall the method introduced in the previous section, and state the result. However, there are some authors that have worked on the geometry of these configurations: for instance, see [ATZ02] or [Shi00].

Let C be any smooth projective curve. $j\colon C \to \mathbb{P}^1$ is said to be a *[3-2]-map* of type $\{n_1, \dots, n_s\}$ if:

- $d = \deg(j)$ is multiple of 12;
- j has multiplicity 3 at each point in $j^{-1}(0)$;
- j has multiplicity 2 at each point in $j^{-1}(1)$;
- there are exactly s points $c_1, \dots, c_s \in \mathbb{P}^1$ in the preimage $j^{-1}(\infty)$, and j has multiplicity n_i at c_i for each i.

This definition, introduced in [MP89], is ad hoc to only consider maps $\mathbb{P}^1 \to \mathbb{P}^1$ that have the potential to be the j-map of our elliptic K3 surface. To abbreviate things, we will say that an s-tuple $\{n_1, \dots, n_s\}$ with $\sum n_i = 24$ *exists* if there is a semistable elliptic K3 surface X with singular fibers $I_{n_1}, \dots, I_{n_s}$.

The next theorem is the counterpart of Theorem 3.2.7:

Theorem 3.3.1. *Let $\{n_1, \dots, n_s\}$ be an s-tuple with $\sum n_i = 24$, and suppose that there exists a [3-2]-map $j\colon \mathbb{P}^1 \to \mathbb{P}^1$ of type $\{n_1, \dots, n_s\}$. Then the configuration $\{n_1, \dots, n_s\}$ exists.*

The proof is identical to the proof of Theorem 3.2.7, and it can be found in [MP89].

As in the rational case, we want to reduce the problem of finding an appropriate j-map to the easier task of finding appropriate monodromy permutations. First, let us fix some notations.
Let $j\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a [3-2]-map of type $\{n_1, \dots, n_s\}$, and let $\{0, 1, \infty, P_1, \dots, P_k\}$ be its branch points.

Choose any P_0 away from the branch points, and consider the loops $\alpha_0, \alpha_1, \alpha_\infty, \beta_1, \ldots, \beta_k$ around the branch points such that their product is the identity of $\pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty, P_1, \ldots, P_k\}, P_0)$. Then the monodromy representation

$$\pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty, P_1, \ldots, P_k\}, P_0) \longrightarrow S_{24}$$

sends these loops respectively into $\sigma_0, \sigma_1, \sigma_\infty, \tau_1, \ldots, \tau_k$. These permutations have the identity as product, they generate a transitive subgroup of S_{24}, and because of our assumptions, σ_0 is forced to have cycle structure 3^8, σ_1 to have cycle structure 2^{12} and σ_∞ to have cycle structure $n_1, \ldots, n_s$.

Not surprisingly, we have:

Theorem 3.3.2. *Let $\{n_1, \ldots, n_s\}$ be an s-tuple with $\sum n_i = 24$. Suppose that there exist permutations $\sigma_0, \sigma_1, \sigma_\infty, \tau_1, \ldots, \tau_k \in S_{24}$ such that they have the identity as product, they generate a transitive subgroup of S_{24}, σ_0 has cycle structure 3^8, σ_1 has cycle structure 2^{12}, σ_∞ has cycle structure $n_1, \ldots, n_s$, and*

$$\sum_{i=1}^{k} \sum_{j} (l_{ij} - 1) = s - 6,$$

where l_{ij} are the lengths of the cycles of τ_i. Then $\{n_1, \ldots, n_s\}$ exists.

Proof. In order to apply the previous theorem, we have only to show that these permutations induce a [3-2]-map $j \colon \mathbb{P}^1 \to \mathbb{P}^1$, i.e. the domain C has genus 0. This is assured by the last condition: the cycle structure of the τ_i's corresponds to the multiplicities of the preimages of $P_1, \ldots, P_k$, and by Hurwitz's formula

$$2g - 2 = -2 \cdot 24 + 8 \cdot (3 - 1) + 12 \cdot (2 - 1) + \sum_{i=1}^{s} (n_i - 1) + \sum_{i=1}^{k} \sum_{j} (l_{ij} - 1),$$

hence $g = 0$. □

In this setting, Table 3.9 can be rephrased as follows:

Proposition 3.3.3. *Suppose that the s-tuple $\{n_1, \ldots, n_s\}$ exists. Then the $(s + 1)$-tuple*

$$\{n_1, \ldots, n_{i-1}, a, b, n_{i+1}, \ldots, n_s\},$$

where $a + b = n_i$, exists.

Proof. Using Table 3.9 we can deform the I_{n_i} fiber into an I_a and an I_b fiber (at the cost of introducing a new transposition τ just before τ_1). Now σ_0 contains only cycles with length multiple of 3, while σ_1 contains only cycles with length multiple of 2, thus we can deform these cycles (using again Table 3.9) in order to force σ_0 to have cycle structure 3^8 and σ_1 to have cycle structure 2^{12}. Now we can apply the previous theorem. □

As it was in the rational case, this theorem is a powerful tool to prove the existence of many configurations. However, we will be using this (actually, the contrapositive) even to eliminate some impossible s-tuples.

Quite interestingly, these two results (Theorem 3.3.2 and Proposition 3.3.3) are again sufficient to prove the existence of all the possible configurations: we find explicit monodromy permutations to show the existence of some s-tuples, and then we deform them to obtain the other ones. Since this can be easily done by a computer, we leave the details to the interested reader; the complete list can be found in [MP89].

Now onto the interesting part: we are going to prove that 135 configurations out of the initial 1242 do not exist. Our strategy will first to introduce the necessary techniques, and then apply them in order to exclude the desired possibilities.

Let $\pi \colon X \to \mathbb{P}^1$ be an elliptic K3 surface with s singular fibers of types $I_{n_1}, \dots, I_{n_s}$. The sublattice R of $\mathrm{NS}(X)$ has rank

$$\sum_i (n_i - 1) = 24 - s;$$

if as usual U denotes the rank 2 unimodular lattice generated by the zero section S_0 and tha class F of a fiber, then the direct sum $A = R \oplus^{\perp} U$ (orthogonality is intended as inside $H^2(X, \mathbb{Z})$) is a sublattice of $\mathrm{NS}(X)$ of rank $26 - s$. Recall that the Mordell-Weil group $\mathrm{MW}(X)$ is isomorphic to the quotient $\mathrm{NS}(X)/A$; in particular, the torsion part $\mathrm{TMW}(X)$ of the Mordell-Weil group is isomorphic to the torsion in $\mathrm{NS}(X)/A$, that is $A^{\perp\perp}/A$: indeed, the quotient $\mathrm{NS}(X)/A^{\perp\perp}$ is torsion-free, and the group $A^{\perp\perp}/A$ is finite. By Table 2.10 we have that

$$G_A \cong G_R \cong \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_s\mathbb{Z},$$

since U is unimodular; moreover the overlattice $A^{\perp\perp}$ of A corresponds to a totally isotropic subgroup $H < G_R = G_A$, i.e.

$$\mathrm{TMW}(X) = A^{\perp\perp}/A = H.$$

Finally, recall that the quadratic form q_R on G_R is given by

$$q_R(x_1 \pmod{n_1}, \dots, x_s \pmod{n_s}) = \sum_{i=1}^{s} \frac{1 - n_i}{2n_i} x_i^2 \pmod{\mathbb{Z}}.$$

Now we are ready to state the new results, the *Length Criterion* and the *Discriminant Criterion*; we will say that the *p-length* of a finite abelian group is the minimum number of generators of its p-Sylow subgroup.

Proposition 3.3.4 (Length Criterion)**.** *Assume that $\pi \colon X \to C$ realizes $\{n_1, \dots, n_s\}$, i.e. it has s singular fibers of types $I_{n_1}, \dots, I_{n_s}$, and fix a prime p. If p divides $s - 3$ or more of the n_i's, then* $\mathrm{TMW}(X)$ *contains non-trivial p-torsion.*

Proof. If p divides $s - 3$ or more of the n_i's, then the p-length of $G_R = \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_s\mathbb{Z}$ is at least $s-3$. Now suppose by contradiction that the p-length of $H = \mathrm{TMW}(X)$ is 0; by Corollary 1.4.8 we have $G_{A^\perp} = G_{A^{\perp\perp}}$, and Remark 1.4.6 shows that $|G_{A^\perp}| = |G_{A^{\perp\perp}}| = \frac{|G_R|}{|H|^2}$, therefore the p-length of $G_{A^\perp}$ equals the p-length of G_R, which is at least $s - 3$. But this is absurd, since $A^\perp$ has rank

$$\mathrm{rk}(H^2(X, \mathbb{Z})) - \mathrm{rk}(A) = 22 - (26 - s) = s - 4,$$

thus the length (and consequently the p-length) of $G_{A^\perp}$ is at most $s - 4$. $\square$

The *Discriminant Criterion* is a refinement of the Length Criterion in a special case:

Proposition 3.3.5 (Discriminant Criterion)**.**

1. *Assume that exactly 4 of $n_1, \dots, n_s$ are odd, say $n_1, \dots, n_4$. Assume further that 4 divides the other numbers $n_5, \dots, n_s$. If*

$$(-1)^s n_1 n_2 n_3 n_4 \not\equiv \prod_{i=5}^{s} (n_i - 1) \pmod{8},$$

then $\mathrm{MW}(X)$ *contains non-trivial 2-torsion.*

2. *Let $p > 2$ be a prime, and assume that exactly $s - 4$ of the n_i are divisible by p, say $n_5, \dots, n_s$. If the product $n_1 n_2 n_3 n_4$ is not a square modulo p, then* $\mathrm{MW}(X)$ *contains non-trivial p-torsion.*

Proof. Let p be any prime, and write $n_i = m_i p^{e_i}$ for $i = 5, \ldots, s$, with $p \nmid m_i$. Assume that $\mathrm{MW}(X)$ has no non-trivial p-torsion. As above

$$|G_{A^\perp}| = |G_{A^{\perp\perp}}| = \frac{|G_R|}{|H|^2} = \frac{1}{|H|^2} \prod_{i=1}^{s} n_i,$$

and by assumption $p \nmid |H| = |\operatorname{TMW}(X)|$. The discriminant of $A^\perp$ equals $|G_{A^\perp}|$ up to a sign; however, since the lattice $H^2(X, \mathbb{Z})$ has signature $(3, 19)$ (this is by the Thom-Hirzebruch topological index theorem, see [BPV84, Theorem I.3.1]) and A has signature $(1, 25 - s)$, necessarily $A^\perp$ has signature $(2, s - 6)$, so there exist $s - 6$ negative eigenvalues for the form on $A^\perp$, and

$$\operatorname{disc}(A^\perp) = (-1)^{s-6} \frac{1}{|H|^2} \prod_{i=1}^{s} n_i = (-1)^{s} \frac{1}{|H|^2} \prod_{i=1}^{s} n_i.$$

In the following, if G is a finite abelian group, we will denote by $G^{(p)}$ its p-part. p does not divide $|H|$, thus

$$G^{(p)}_{A^{\perp\perp}} = G^{(p)}_{A^\perp} = G^{(p)}_R = G^{(p)}_A = \mathbb{Z}/p^{e_5}\mathbb{Z} \times \ldots \times \mathbb{Z}/p^{e_5}\mathbb{Z};$$

moreover the induced quadratic form $q^{(p)}_{A^\perp}$ on $G^{(p)}_{A^\perp}$ is

$$\begin{aligned} q^{(p)}_{A^\perp}(x_5 \pmod{p^{e_5}}, \ldots, x_s \pmod{p^{e_s}}) &= -q^{(p)}_{A^{\perp\perp}}(x_5 \pmod{p^{e_5}}, \ldots, x_s \pmod{p^{e_s}}) = \\ &= -q_R(0,0,0,0, m_5 x_5 \pmod{n_5}, \ldots, m_s x_s \pmod{n_s}) = \\ &= -\sum_{j=5}^{s} \frac{1 - n_j}{2n_j} m_j^2 x_j^2 = \sum_{j=5}^{s} \frac{(n_j - 1) m_j}{2p^{e_j}} x_j^2, \end{aligned}$$

where we are identifying $\{0\} \times \mathbb{Z}/p^{e_i}\mathbb{Z} < \mathbb{Z}/m_i\mathbb{Z} \times \mathbb{Z}/p^{e_i}\mathbb{Z}$ as the subgroup $m_i\mathbb{Z}/m_i p^{e_i}\mathbb{Z} < \mathbb{Z}/m_i p^{e_i}\mathbb{Z}$. In other words, the form $q^{(p)}_{A^\perp}$ diagonalizes. Passing to the p-adics, i.e. considering the lattice $A^\perp \otimes \mathbb{Z}_p$, we have that the induced bilinear form on $A^\perp \otimes \mathbb{Z}_p$ diagonalizes over $\mathbb{Z}_p$ (see for instance [Nik79]: here we are using the hypotheses on the divisibility, i.e. that 4 divides $n_5, \ldots, n_s$ in the first case, and that p divides $n_5, \ldots, n_s$ in the second), with eigenvalues (relative to an appropriate $\mathbb{Z}_p$-basis)

$$\frac{p^{e_5}}{(n_5 - 1) m_5}, \ldots, \frac{p^{e_s}}{(n_s - 1) m_s}$$

since the bilinear form differs from the quadratic form by a multiplication by 2. These entries are well defined up to square of units in $\mathbb{Z}_p$ (again, see [Nik79]), hence we can change basis and obtain a diagonal matrix for the form on $A^\perp \otimes \mathbb{Z}_p$ given by

$$p^{e_5}(n_5 - 1) m_5, \ldots, p^{e_s}(n_s - 1) m_s$$

in the diagonal. Consequently, the discriminant

$$\operatorname{disc}(A^\perp \otimes \mathbb{Z}_p) = \prod_{j=5}^{s} n_j (n_j - 1)$$

must be equal (up to square of units in $\mathbb{Z}_p$) to the discriminant

$$\operatorname{disc}(A^\perp) = (-1)^{s} \frac{1}{|H|^2} \prod_{i=1}^{s} n_i.$$

Since by assunption $|H|$ is a unit in $\mathbb{Z}_p$, we have that

$$(-1)^s n_1 n_2 n_3 n_4 \equiv \prod_{j=5}^{s} (n_j - 1) \pmod{(\mathbb{Z}_p^*)^2}.$$

Now, if $p = 2$ we are done, since equality modulo squares in $\mathbb{Z}_2$ is measured by the residue classes modulo 8; if instead $p > 2$, equality modulo squares in $\mathbb{Z}_p$ is measured by equality modulo squares in $\mathbb{Z}/p\mathbb{Z}$, so we are done again (notice that $n_j - 1 \equiv -1 \pmod p$ for all $j \geq 5$). □

As we remarked previously, we can interpret a torsion section S as an s-tuple $(k_1 \pmod{n_1}, \dots, k_s \pmod{n_s}) \in H$ that is isotropic for the quadratic form q_R. If we label the components of each singular fiber going around the cycle (putting as the zero component the one that intersects the zero section S_0), we have seen that S intersects the k_j^{th} component of the j^{th} singular fiber. Thus the s-tuple completely determines the section S and its intersections with the fibers.

Now we are going to see how to use the existence of p-torsion in $\mathrm{MW}(X)$ to prove the impossibility of some configurations. As above, let $\pi\colon X \to \mathbb{P}^1$ be a semistable elliptic K3 surface, fix a prime p, and let S be a p-torsion section in $\mathrm{TMW}(X)$. The translation τ_S by S induces an automorphism τ_S of X of order precisely p; moreover, the quotient $Y = X/\tau_S$ is again a (possibly singular) elliptic K3 surface. Clearly it is an elliptic surface: τ_S restricts to an automorphism of the generic fiber, and its quotient is again a smooth elliptic curve; we will denote by $\pi'\colon Y \to \mathbb{P}^1$ the elliptic fibration. Moreover it is a K3 surface: τ_S preserves the nowhere-vanishing holomorphic 2-form on X, which descends to a well-defined nowhere-vanishing 2-form on Y (and the irregularity cannot increase).

Our goal is to describe the singular fibers of the new fibration $\pi'\colon Y \to \mathbb{P}^1$; notice that τ_S has no fixed points on the smooth fibers, and the only singular fibers on Y come from singular fibers on X. In the next proposition we fix an I_n fiber on X.

Proposition 3.3.6. *Label the components of the I_n fiber as above; let them be $C_0, \dots, C_{n-1}$.*

1. *If the p-torsion section S meets C_0, then τ_S fixes the nodes of the I_n fiber and no other point. Locally, near each node, the action of τ_S can be linearized to be $(x, y) \mapsto (\zeta_p x, \zeta_p^{-1} y)$, where $\zeta_p = \exp(\frac{2\pi i}{p})$ is a primitive p^{th} root of unity. In particular, by Proposition 1.3.20, the quotient Y has an A_{p-1} singularity corresponding to each node of the I_n fiber.*

2. *If S does not meet C_0, then $p \mid n$ and S meets $C_{k\frac{n}{p}}$ for some k. In particular τ_S has no fixed points on the I_n fiber.*

Proof. 1. The smooth points F_{ns} of the I_n fiber are an extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{C}^*$, i.e. they fit into an exact sequence of groups

$$0 \longrightarrow \mathbb{C}^* \longrightarrow F_{ns} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0;$$

τ_S acts on F_{ns} by multiplication by $S \cap C_0$. Identifying the smooth points of C_0 with $\mathbb{C}^*$, the translation τ_S must act on them as an automorphism of order exactly p, i.e. it is a primitive p^{th} root of unity, that we can assume to be ζ_p. To make this identification coherent on all the components, we choose a local coordinate on C_0 such that $0 \in \mathbb{P}^1$ corresponds to $C_0 \cap C_{n-1}$, $\infty \in \mathbb{P}^1$ corresponds to $C_0 \cap C_1$, and $1 \in \mathbb{C}^* \subseteq \mathbb{P}^1$ corresponds to $S_0 \cap C_0$; repeating this choice for all the other components, we can assume that τ_S acts as ζ_p on all subsets of smooth points of each component. Clearly, the only fixed points under the action of τ_S are the nodes of the fiber. Now, we have chosen coordinates x, y for the two intersecting components such that the node looks locally as $\{xy = 0\}$, and the directions x, y are exactly the eigenspaces for τ_S with eigenvalues respectively ζ_p and ζ_p^{-1} (since $0 \in C_{i-1}$ corresponds to $\infty \in C_i$).

2. Reasoning as in the first point, we have that τ_S acts on F_{ns} by multiplication by $C_i \cap S$, where C_i is the component of the fiber meeting S. Therefore τ_S moves C_0 onto C_i. Since τ_S has order p, necessarily $ip \equiv 0 \pmod n$, i.e. $i = k\frac{n}{p}$ for some k (invertible modulo p). In other words, τ_S rotates the fiber, and in particular it has no fixed points.

□

Corollary 3.3.7. *Keep the notations as in the previous proposition.*

1. *If $p \nmid n$, then τ_s fixes the nodes of the I_n fiber, and Y has an I_{pn} fiber underneath the I_n fiber.*
2. *If $p \mid n$ and S meets C_0, then τ_S fixes the nodes of the I_n fiber, and Y has an I_{pn} fiber underneath the I_n fiber.*
3. *If $p \mid n$ and S does not meet C_0, then τ_S has no fixed points on the I_n fiber, and Y has an $I_{\frac{n}{p}}$ fiber underneath the I_n fiber.*

Proof. Everything is fairly easy: in the first two cases we apply the first point of the previous proposition, and we recall that an A_{p-1} singularity is resolved by a string of $(p-1)$ (-2)-curves. In the last case we apply the second point of the proposition, and we see that τ_S identifies C_i with $C_{i+k\frac{n}{p}}$ for every i: the cycle of n rational curves descends to a cycle of $\frac{n}{p}$ rational curves. □

From this we obtain in particular that Y has the same number of singular fibers as X, and it is again semistable. The next lemma is particularly useful to limit the possibilities for the order p of our torsion section S.

Lemma 3.3.8. *Let τ be an automorphism of order p of a K3 surface X, with a finite number of fixed points, and such that the quotient $Y = X/\tau$ is again a K3 surface. Assume further that each fixed point of τ gives an A_{p-1} singularity on Y; let $\widetilde{Y}$ be the resolution of the singularities of Y. Then the number of fixed points of τ is $\frac{24}{p+1}$.*

Proof. Let $x_1, \dots, x_k \in X$ be the fixed points, and let $y_1, \dots, y_k \in Y$ be their images in the quotient. Then the projection $X \to Y$ is a covering of degree p outside the points $x_1, \dots, x_k$ and $y_1, \dots, y_k$, hence

$$24 = \chi_{top}(X) = \chi_{top}(X \backslash \{x_1, \dots, x_k\}) + k = p \cdot \chi_{top}(Y \backslash \{y_1, \dots, y_k\}) + k.$$

Now Y and $\widetilde{Y}$ are isomorphic outside $y_1, \dots, y_k$ and the corresponding A_{p-1} curves; moreover, removing an A_{p-1} curve from $\widetilde{Y}$ decreases the Euler characteristic of $\widetilde{Y}$ by p, hence

$$24 = p \cdot \chi_{top}(Y \backslash \{y_1, \dots, y_k\}) + k = p(24 - kp) + k,$$

since $\widetilde{Y}$ is again a K3 surface. From this it is immediate to check that $k = \frac{24}{p+1}$. □

This forces p to be one of $2, 3, 5, 7, 11$. Actually, we can do better:

Corollary 3.3.9. *Let S be a p-torsion section with at least one fixed point. Then $p = 2, 3, 5, 7$, and τ_S has respectively $8, 6, 4, 3$ fixed points.*

Proof. We have only to exclude $p = 11$. If this were the case, $\widetilde{Y}$ would contain two A_{10} curves; this is impossible since these are 20 distinct classes in $\mathrm{NS}(X)$, and distinct from the class of the zero section, yielding $\rho > 20$, a contradiction. □

This can be applied to our semistable elliptic K3 surface thanks to the following:

Corollary 3.3.10 (Fixed Point Rule)**.** *Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic K3 surface with configuration of the fibers given by $\{n_1, \dots, n_s\}$. Assume that there is non-trivial p-torsion in* $\mathrm{MW}(X)$. *Then there exists a subset of $\{n_1, \dots, n_s\}$ of integers divisible by p that sum to $\frac{24p}{p+1}$.*

Proof. Let S be our p-torsion section. Consider the subset $S \subseteq \{n_1, \dots, n_s\}$ formed by the n_i's such that S does not meet the zero component of the I_{n_i} fiber; S contains only multiples of p by Corollary 3.3.7. Since $\sum n_i = 24$ and the sum of the integers in $\{n_1, \dots, n_s\} \backslash S$ is $\frac{24}{p+1}$ by the previous result, necessarily the integers in S sum to $24 - \frac{24}{p+1} = \frac{24p}{p+1}$. □

Now we are ready to prove the non-existence of the 135 impossible configurations, using the techniques just introduced.

Proposition 3.3.11. *The following* 3 9*-tuples do not exist:*

$$\{1,2,2,2,2,2,2,2,9\}, \quad \{2,2,2,2,2,2,2,3,7\}, \quad \{2,2,2,2,2,2,2,5,5\}$$

Proof. By the Length Criterion, if any of these 3 configurations existed, $\mathrm{MW}(X)$ would contain 2-torsion; however this is impossible by the Fixed Point Rule, since $\frac{24\cdot 2}{3} = 16$ and these configurations contain at most 7 2's. □

Proposition 3.3.12. *The following* 11 8*-tuples do not exist:*

$$\begin{array}{llll} \{1,1,1,4,4,4,4,5\}, & \{1,1,3,3,3,3,3,7\}, & \{1,2,2,2,2,2,2,11\}, & \{1,2,2,2,2,2,4,9\}, \\ \{1,3,3,3,3,3,4,4\}, & \{2,2,2,2,2,2,2,10\}, & \{2,2,2,2,2,2,3,9\}, & \{2,2,2,2,2,2,5,7\}, \\ \{2,2,2,2,2,3,4,7\}, & \{2,2,2,2,2,4,5,5\}, & \{2,2,3,3,3,3,3,5\} & \end{array}$$

Proof. Using Proposition 3.3.3 (in fact, its contrapositive) and the previous result, we eliminate 7 out of the 11 listed configurations. We remain with

$$\{1,1,1,4,4,4,4,5\}, \quad \{1,1,3,3,3,3,3,7\}, \quad \{1,3,3,3,3,3,4,4\}, \quad \{2,2,3,3,3,3,3,5\}.$$

Assume the first one exists. By the Discriminant Criterion, $\mathrm{MW}(X)$ has non-trivial 2-torsion, say S. The translation τ_S has 8 fixed points, hence all these 8 points come from the 3 I_1 fibers and the I_5 fiber. Consequently S can be identified with $(0,0,0,2,2,2,2,0) \in G_R$. Therefore the quotient of X by τ_S has configuration $\{2,2,2,2,2,2,2,10\}$, of which we have proved the impossibility.
The other 3 configurations are easier to rule out: if they existed, $\mathrm{MW}(X)$ would contain non-trivial 3-torsion by the Length Criterion, contradicting the Fixed Point Rule. □

Proposition 3.3.13. *The following* 34 7*-tuples do not exist:*

$$\begin{array}{lllll} \{1,1,1,4,4,4,9\}, & \{1,1,1,4,4,5,8\}, & \{1,1,3,3,3,3,10\}, & \{1,1,3,3,3,6,7\}, & \{1,1,3,4,5,5,5\}, \\ \{1,1,4,4,4,4,6\}, & \{1,1,4,4,4,5,5\}, & \{1,2,2,2,2,2,13\}, & \{1,2,2,2,2,4,11\}, & \{1,2,2,2,2,6,9\}, \\ \{1,2,2,2,4,4,9\}, & \{1,2,3,3,5,5,5\}, & \{1,2,4,4,4,4,5\}, & \{1,3,3,3,3,3,8\}, & \{1,3,3,3,3,4,7\}, \\ \{1,3,3,3,4,4,6\}, & \{2,2,2,2,2,2,12\}, & \{2,2,2,2,2,3,11\}, & \{2,2,2,2,2,4,10\}, & \{2,2,2,2,2,5,9\}, \\ \{2,2,2,2,2,7,7\}, & \{2,2,2,2,3,4,9\}, & \{2,2,2,2,3,6,7\}, & \{2,2,2,2,4,5,7\}, & \{2,2,2,2,5,5,6\}, \\ \{2,2,2,3,4,4,7\}, & \{2,2,2,4,4,5,5\}, & \{2,2,3,3,3,3,8\}, & \{2,2,3,3,3,5,6\}, & \{2,3,3,3,3,3,7\}, \\ \{2,3,3,3,3,5,5\}, & \{2,3,3,4,4,4,4\}, & \{3,3,3,3,3,4,5\}, & \{3,3,3,3,4,4,4\} & \end{array}$$

Proof. Using the deformation result (Proposition 3.3.3) and the previous result, we exclude 31 of the 34 listed configurations, remaining with

$$\{1,1,3,4,5,5,5\}, \quad \{1,2,3,3,5,5,5\}, \quad \{2,3,3,4,4,4,4\}.$$

The first two are impossible: $\mathrm{MW}(X)$ must contain non-trivial 5-torsion by the Discriminant Criterion, violating the Fixed Point Rule. Assume the last one exists. By the Fixed Point Rule, $\mathrm{MW}(X)$ can only contain 2-torsion. By the Length Criterion, $\mathrm{TMW}(X)$ contains a section S of order exactly 2; since S has 8 fixed points, necessarily S can be identified with $x = (0,0,0,2,2,2,2) \in G_R$. S is the only element in $\mathrm{TMW}(X)$ with order 2 (since the other isotropic elements of order 2 correspond to torsion sections with not 8 fixed points), hence $\mathrm{TMW}(X)$ is cyclic.
Assume that $\mathrm{TMW}(X)$ has order 2, i.e. it only contains S and the zero section S_0. Let $H < G_R$ be the isotropic subgroup isomorphic to $\mathrm{TMW}(X)$; then a 7-tuple $y = (a_1, \ldots, a_7) \in G_R$ belongs to $H^\perp$ if and only if $\langle x, y \rangle = q_R(x+y) - q_R(x) - q_R(y) = q_R(x+y) - q_R(y)$ is an integer, i.e. if and only if

$$\frac{3}{8}\left[4\sum_{i=4}^{7} a_i + 16\right] \in \mathbb{Z},$$

that is $\sum_{i=4}^{7} a_i \equiv 0 \pmod 2$. Consequently $H^\perp$ is isomorphic to

$$\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^3,$$

and $H = \langle x \rangle$ is contained inside one of the $\mathbb{Z}/4\mathbb{Z}$ (indeed, there exists the element $x' = (0,0,0,1,1,1,1) \in H^\perp$ of order 4 such that $2x' = x$), hence

$$H^\perp/H \cong (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/3\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^2.$$

This is absurd: the length of $H^\perp/H$ is 5, while $A^\perp$ has rank $3 < 5$.
Therefore $\mathrm{TMW}(X)$ must have order ≥ 4, and there exists an element x' of order 4 such that $2x' = x$. x' is one of the elements $(0 \text{ or } 1, 0, 0, \pm1, \pm1, \pm1, \pm1)$ of G_R. However none of these elements is isotropic, and in particular it cannot represent a section in $\mathrm{TMW}(X)$. □

Proposition 3.3.14. *The following* 87 6*-tuples do not exist:*

$$\begin{array}{lllll}
\{1,1,1,1,8,12\}, & \{1,1,1,3,9,9\}, & \{1,1,1,4,4,13\}, & \{1,1,1,4,5,12\}, & \{1,1,1,4,8,9\}, \\
\{1,1,1,5,8,8\}, & \{1,1,1,6,6,9\}, & \{1,1,2,2,8,10\}, & \{1,1,2,6,7,7\}, & \{1,1,3,3,3,13\}, \\
\{1,1,3,3,6,10\}, & \{1,1,3,3,7,9\}, & \{1,1,3,4,5,10\}, & \{1,1,3,5,5,9\}, & \{1,1,3,6,6,7\}, \\
\{1,1,4,4,4,10\}, & \{1,1,4,4,5,9\}, & \{1,1,4,4,6,8\}, & \{1,1,4,5,5,8\}, & \{1,1,5,5,5,7\}, \\
\{1,2,2,2,2,15\}, & \{1,2,2,2,4,13\}, & \{1,2,2,2,6,11\} & \{1,2,2,2,8,9\}, & \{1,2,2,3,8,8\}, \\
\{1,2,2,4,4,11\}, & \{1,2,2,4,6,9\}, & \{1,2,2,5,7,7\}, & \{1,2,3,3,5,10\}, & \{1,2,3,4,7,7\}, \\
\{1,2,3,5,5,8\}, & \{1,2,3,6,6,6\}, & \{1,2,4,4,4,9\}, & \{1,2,4,4,5,8\}, & \{1,2,5,5,5,6\}, \\
\{1,3,3,3,3,11\}, & \{1,3,3,3,4,10\}, & \{1,3,3,3,6,8\}, & \{1,3,3,3,7,7\}, & \{1,3,3,4,4,9\}, \\
\{1,3,3,4,6,7\}, & \{1,3,3,5,5,7\}, & \{1,3,4,4,6,6\}, & \{1,3,4,5,5,6\}, & \{1,3,5,5,5,5\}, \\
\{1,4,4,4,4,7\}, & \{1,4,4,4,5,6\}, & \{1,4,4,5,5,5\}, & \{2,2,2,2,2,14\}, & \{2,2,2,2,3,13\}, \\
\{2,2,2,2,4,12\}, & \{2,2,2,2,5,11\}, & \{2,2,2,2,6,10\}, & \{2,2,2,2,7,9\}, & \{2,2,2,3,4,11\}, \\
\{2,2,2,3,6,9\}, & \{2,2,2,3,7,8\}, & \{2,2,2,4,4,10\}, & \{2,2,2,4,5,9\}, & \{2,2,2,4,7,7\}, \\
\{2,2,2,5,5,8\}, & \{2,2,2,5,6,7\}, & \{2,2,3,3,3,11\}, & \{2,2,3,3,5,9\}, & \{2,2,3,3,6,8\}, \\
\{2,2,3,4,4,9\}, & \{2,2,3,4,6,7\}, & \{2,2,3,5,6,6\}, & \{2,2,4,4,5,7\}, & \{2,2,4,5,5,6\}, \\
\{2,3,3,3,3,10\}, & \{2,3,3,3,5,8\}, & \{2,3,3,3,6,7\}, & \{2,3,3,4,4,8\}, & \{2,3,3,5,5,6\}, \\
\{2,3,4,4,4,7\}, & \{2,3,4,5,5,5\}, & \{2,4,4,4,4,6\}, & \{2,4,4,4,5,5\}, & \{3,3,3,3,3,9\}, \\
\{3,3,3,3,4,8\}, & \{3,3,3,3,5,7\}, & \{3,3,3,4,4,7\}, & \{3,3,3,4,5,6\}, & \{3,3,3,5,5,5\}, \\
\{3,3,4,4,4,6\}, & \{3,4,4,4,4,5\} & & &
\end{array}$$

Proof. Using again the deformation result and the non-existence of the previous 7-tuples, we remove 78 of the 87 listed 6-tuples, remaining with

$$\begin{array}{lllll}
\{1,1,1,1,8,12\}, & \{1,1,1,3,9,9\}, & \{1,1,1,6,6,9\}, & \{1,1,2,2,8,10\}, & \{1,1,2,6,7,7\}, \\
\{1,2,2,3,8,8\}, & \{1,2,2,5,7,7\}, & \{1,2,3,4,7,7\}, & \{1,2,3,6,6,6\}. &
\end{array}$$

However, they are all easy to rule out. For the first one, the Discriminant Criterion forces a non-trivial 2-torsion section, that cannot exist by the Fixed Point Rule. For the second and third ones, the Length Criterion forces a non-trivial 3-torsion section; this is a contradiction for the third one (using the Fixed Point Rule). Instead, this 3-torsion section in the second configuration must be of the form $(0,0,0,0,\pm3,\pm3)$, since it has 6 fixed points, but then the quotient would have configuration $\{3,3,3,9,3,3\}$, which we have excluded.
For the fourth and the last one, the Length Criterion forces a non-trivial 2-torsion section, violating the Fixed Point Rule. For the 5^{th}, 7^{th} and 8^{th} ones, the Discriminant Criterion forces a non-trivial 7-torsion section, violating again the Fixed Point Rule.
Finally, for the sixth one, the Length Criterion forces a non-trivial 2-torsion section, which must be $(0,0,0,0,4,4)$, since it has 8 fixed points. However, the quotient would have configuration $\{2,4,4,6,4,4\}$, which we have excluded. □

Assuming that we have checked the existence of all the other 1107 configurations as explained at the beginning of the section, we have:

Theorem 3.3.15. *There exist exactly* 1107 *configurations of semistable singular fibers for an elliptic K3 surface, which are those s-tuples $\{n_1, \ldots, n_s\}$ with $s \geq 6$ and $\sum n_i = 24$ not listed above.*
In particular, all the s-tuples with $s \geq 10$ exist.

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